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► **To cite this version:**

Florian Navarro. Cooperative games with types, outside options and the egalitarian value. 2024.
hal-04324424v3

HAL Id: hal-04324424

<https://univ-angers.hal.science/hal-04324424v3>

Preprint submitted on 16 Sep 2024

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Cooperative games with types, outside options and the egalitarian value¹

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Abstract

This article introduces a new axiom of sub-games monotonicity for TU-games as well as a model of cooperative games with types. The axiom, alongside efficiency, characterizes the egalitarian value. The model addresses situations where players of different types are needed. Each player has a specific type and coalitions are feasible only if it contains at most one player of each type. We use the new characterization of the egalitarian value for TU-games to obtain the following result in our class of problems: the egalitarian value is the only sharing rule that ensures that each player of the most productive group is better off joining this most productive group. We characterize the egalitarian value without fairness requirement and show that, for this new class of problems, egalitarianism can provide some form of incentives towards optimal cooperation.

Keywords: Cooperative game theory, shapley value, equal division, egalitarian value, type structure, incentives

1. Introduction

In the present article we develop a model of cooperative games in which we need players of different types. Several players are of the same type, but a player can only be of a single type. The type structure restricts the feasibility

¹I would like to thank Sylvain Béal who was always available to provide insightful feedbacks and comments at various stages of this paper. I also thank Stéphane Gonzalez for seeing some merits in the original draft, which gave me the confidence to pursue this idea.

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of coalitions: a coalition is feasible if it contains at most one player of each type. This means we have situations in which the grand coalition cannot form. Such a situation can arise when members of an organization have to form a subgroup to tackle a specific project (see example 3). Several groups having to choose a representative to participate in a committee can also be a situation covered by our model. The problem raised by our model is thus different than the usual problem of cooperative games (sharing the worth of the grand coalition).

A solution ϕ to this model consists of two mechanisms: a function μ that selects a group of players specifying, for each type, a single player of this type, and a sharing rule ψ that shares the worth produced by the group amongst its members. We are interested in a solution that fosters the emergence of the most productive group with all types. To this end, we propose an axiom of *optimality* (the solution selects the best group) on μ and an axiom of *no outside options* (no member of the selected group can be better off in another group) that apply to the solution ϕ . Together, these axioms will impose requirements on the solution ψ that is applied to the allocation problem for the chosen group. We introduce an axiom of *sub-games monotonicity* for the sharing rule ψ . This axiom asks that no member of a population can be worse off by choosing to associate with a group of a given size with which it has the best synergy. We show that requiring the axiom of *no outside option* on ϕ for all μ that satisfy *optimality* is equivalent to requiring *sub-games monotonicity* on ψ . As an intermediary result, we offer a characterization of the egalitarian value (which shares the total worth equally between all players) that relies on the *sub-games monotonicity* axiom alongside the standard axiom of *efficiency*. As such we characterize the egalitarian principle without any fairness criteria.

Formally, our model is related to the Aumann-Drèze model ([Aumann and Dreze, 1974](#)) as it also involves coalition structures. In a coalition structure, players are part of pre-existing groups that can not communicate with each others. This structure has an effect on the worth of coalitions and alter the characteristic function of the underlying transferable utility game (TU-game). [Owen \(1977\)](#) also relies on coalition structures (called a-priori unions) although

they directly affect the payoff of the players. In the present article we take yet another direction: this structure restricts the formation of coalitions and therefore their feasibility.

More generally, this paper joins a line of works which enrich the standard model of cooperative game theory by considering new structures of cooperation. A large part of this research focuses on structures that restrict cooperation by modelling either asymmetries in communication, or hierarchical constraints. In two papers, Myerson modelled communication by respectively a graph (1977) and an hyper-graph (1980). Both represent a restriction of communication between players. Various solution concepts have been proposed for such games. We can cite for instance the Position-value (Meessen, 1988), the Average-Tree Value (Herings et al., 2008), the Mean value (Hamiache, 2004), the F-value (Hamiache, 1999), the Hamiache-Navarro value (Hamiache and Navarro, 2019) and numerous variations of the Myerson value³. Additionally, different forms of hierarchical constraints have been explored by Gilles et al. (1992), Faigle and Kern (1992) and more recently by Béal et al. (2021). The distinction between the framework of Aumann and Dreze (1974), Owen (1977) and ours can similarly be made for, respectively, the framework of Gilles et al. (1992), Béal et al. (2021) and Faigle and Kern (1992). Béal et al. (2021) offer a deeper discussion on how the same mathematical model (in their case, of hierarchical constraint) can affect the underlying game in different ways. In this sense, our work is in line with this trend of research.

Another approach to types, or more precisely differently skilled players, has been proposed by Bachrach et al. (2013) with coalitional skill games (CSG). However, in a CSG framework, certain players can perform certain tasks and the worth of a coalition depends on the nature or number of tasks that can be performed by the coalition. The model is thus similar to the standard model of cooperative game theory but restricts itself to a certain type of characteristic

³For instances two efficient extensions of the Myerson value were proposed by Hamiache (2012) and Béal et al. (2015). Examples of variations on the theme can be found by Gómez et al. (2003) and González-Arangüena et al. (2003).

functions defined by the players’ “skills”. Therefore, to the best of our knowledge few models have been proposed to enrich the individual characteristics of the players.

Our model tackles issues not addressed by existing research and contributes to the economic theory literature on several levels. For the standard model of TU-games, we provide a new characterization of the egalitarian value. In addition, we use a formal framework of coalition structure to develop a novel class of unexplored problems as of yet. Within this class, we investigate the effect of sharing rules on the spontaneous emergence of coalitions, making it somewhat endogenous to the model. We give a solution for these problems and show that, for this class of problems, egalitarianism can provide incentives whereas approaches based on the evaluation of marginal contributions can be detrimental for cooperation.

In section 2 of this paper we introduce the canonical model of cooperative games with transferable utility and present the axiom of *sub-games monotonicity* with two variations (a stronger one and a weaker one). We build a new characterization of the egalitarian value and study how this characterization holds when weakening or strengthening the *sub-games monotonicity* axiom. In section 3 we present our new model of games with types and we propose axioms for a desirable solution to this new type of problems as well as present our main results. We conclude in section 4.

2. TU-games, egalitarian value and sub-games monotonicity

Let U be a non-empty and infinite set of players. A coalition is a non-empty and finite subset of U . A coalitional game with transferable utility (also called a cooperative TU-game) is a pair (N, v) where N is a coalition and v is a function satisfying $v : 2^N \rightarrow \mathbb{R}$ and $v(\emptyset) = 0$. We denote by Γ the set of these games. Given a coalition $S \subseteq N$ we write $s = |S|$ its cardinality. For any $S \subsetneq N$ we will write $(S, v|_S)$ the sub-game restricted to coalition S with $v|_S(T) = v(T)$,

$\forall T \subseteq S$. A player $i \in N$ is a null player in the game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. A player $i \in N$ is a nullifying player in the game (N, v) if $v(S) = 0$, for all $S \subseteq N$ such that $i \in S$. Two players $i, j \in N$ are equals in the game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A TU-game v is said to be a convex game if it satisfies $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S \subseteq T \subseteq N \setminus \{i\}$. Convex game have been shown by [Shapley \(1971\)](#) to have a non-empty core⁴.

A sharing rule on Γ is a function ψ which associates with each game $(N, v) \in \Gamma$ a vector $\psi(N, v) \in \mathbb{R}^N$. The Shapley value, first presented by [Shapley \(1953\)](#), rewards players based on their marginal contributions, weighted by all the possible orders of entry of players in coalitions. We write it Sh and define it as

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{i\})], \forall i \in N.$$

The Egalitarian value divides equally the worth of coalition N between all players. We write this sharing rule EG and define it as

$$EG_i(N, v) = \frac{v(N)}{n}, \forall i \in N.$$

We introduce the following classical axioms for sharing rules over TU-games.

Efficiency. For each game $(N, v) \in \Gamma$ it holds that $\sum_{i \in N} \psi_i(N, v) = v(N)$.

Linearity. For each pair of games $(N, v), (N, w) \in \Gamma$ and each real number $\alpha \in \mathbb{R}$, it holds that $\psi(N, v + \alpha w) = \psi(N, v) + \alpha \psi(N, w)$.

Null player. For each game $(N, v) \in \Gamma$ and each null player $i \in N$, it holds that $\psi_i(N, v) = 0$.

Nullifying player. For each game $(N, v) \in \Gamma$ and each nullifying player $i \in N$, it holds that $\psi_i(N, v) = 0$.

Equal treatment of equals. For each game $(N, v) \in \Gamma$ and each couple of players $i, j \in N$ who are equals in (N, v) , it holds that $\psi_i(N, v) = \psi_j(N, v)$.

⁴Quoting this clear and concise definition from [Shapley \(1971\)](#), “the core of [a TU-game] is the set of feasible outcomes that cannot be improved upon by any coalition of players”.

As shown by [Shapley \(1953\)](#), the Shapley value is the unique sharing rule satisfying efficiency, linearity, null player and equal treatment of equals. The egalitarian value has been shown by [van den Brink \(2007\)](#) to be the unique sharing rule satisfying efficiency, linearity, nullifying player and the equal treatment of equals.

In 1983, [Thomson](#) introduced the axiom of population monotonicity in the context of fair division. This axiom stated that when facing a given problem of fair division, no player should be better off by having to share with more players than less. [Chun \(1986\)](#) and later [Chun and Park \(2012\)](#) expanded on this idea and proposed a similar axiom for, respectively, fair division problem and allocation problems in cooperative games. This weaker variant states that when a population is joined by newcomers, all members of the original population are either all better off or all worse off.

We introduce a new axiom of **sub-games monotonicity** that have some conceptual relation to the axioms of population monotonicity introduced by [Thomson \(1983\)](#), [Chun \(1986\)](#) and [Chun and Park \(2012\)](#). Let us consider a given population P offered with the choice of cooperating with two different groups of the same size. Our axiom states that no member of population P can be worse off if the population chooses the group with whom it has the best synergy. In the context of cooperative games, given two coalition of players whose intersection is non-empty, we ask that no member of the intersection is rewarded less when considering the coalition with the better worth. Formally⁵,

Sub-games monotonicity. *For each game (N, v) with $n \geq 3$ and each couple of coalitions $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$ if $v(S) \geq v(T)$ it holds that $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$, $\forall i \in S \cap T$.*

We also give an alternative interpretation for this axiom. If a subgroup P of S decides to replace the other members of S by an identical number of outsiders,

⁵ P corresponds to $S \cap T$ in the definition of sub-game monotonicity

then the members of P should not be worse off if the newly formed coalition T is at least as productive as the original coalition S . Note that this axiom puts no requirement on any payoffs in the game (N, v) . Moreover, for the two sub-games considered, it says nothing about the payoffs of players not belonging to the intersection of S and T . As a result, this axiom is not as strong as it can first appear.

The axiom of *sub-games monotonicity* is also quite different from the axiom of *grand coalition monotonicity* used by [Casajus and Huettner \(2014\)](#) to characterize the egalitarian value. Grand coalition monotonicity compares the same population involved in two different games, whereas we compare two different populations. Nonetheless some proximity can be seen in the sense that our axiom also provides a monotonic approach to the egalitarian principle.

Before diving into the results we provide two motivating examples.

Example 1. We consider the game (N, v) with $N = \{1, 2, 3, 4\}$ and v such that $v(S) = 0$ for all singletons and

S	12	13	14	23	24	34	123	124	134	234	1234
$v(S)$	1	2	1	1	1.5	1	4	3	4	3	6

Observe that (N, v) is convex. We focus on the two coalitions $S = \{1, 2, 3\}$ and $T = \{1, 2, 4\}$. We might want to provide incentives for players 1 and 2 to join with player 3 instead of player 4. However, the Shapley value of the corresponding sub-games are

Sh_i	$v_{ \{1,2,3\}}$	$v_{ \{1,2,4\}}$
1	1.5	$\frac{10}{12}$
2	1	$\frac{13}{12}$
3	1.5	-
4	-	$\frac{13}{12}$
\sum	4	3

Player 1 would prefer to join with 3 and player 2 would prefer to join with 4. The Shapley value is therefore not sub-games monotonic, even for convex game.

■

This second example show that the Shapley value can even be “anti-monotonic” in the sub-games.

Example 2. We consider the game (N, v) with $N = \{1, 2, 3, 4\}$, v such that

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
$v(S)$	0.5	0	0.5	0	1	1.2	0.5	1	0	0.5	2	1	1.2	1	2

Observe that this game is also convex. We focus on the four coalitions of size

2. The Shapley value of the corresponding sub-games are

Sh_i	$v_{\{1,2\}}$	$v_{\{1,3\}}$	$v_{\{2,4\}}$	$v_{\{3,4\}}$
1	0.75	0.6	-	-
2	0.25	-	0	-
3	-	0.6	-	0.5
4	-	-	0	0
Σ	1	1.2	0	0.5

In this example, player 1 and player 2 both have incentives to form $\{1, 2\}$ which has a worth lower than $\{1, 3\}$.

■

Lemma 1 For every game $(N, v) \in \Gamma$ if the sharing rule $\psi(N, v)$ satisfies **efficiency** and **sub-games monotonicity** then it satisfies **equal treatment of equals**.

Proof. We consider a game (N, v) such that two players $i, j \in N$ are equal in v . Let $P \subset U \setminus N$ with $p = n - 1$. We build a larger game (N^+, w) such that $N^+ = N \cup P$. We denote $N_i = P \cup \{i\}$ and $N_j = P \cup \{j\}$. We take w such that

- $w(S) = v(S)$ for all $S \subseteq N$,
- $w(N_i) = w(N_j) = w(N) = v(N)$,

Since U is infinite, such a construction is always possible.⁶

We consider coalition N and we will prove that $\psi_i(N, w|_N) = \psi_j(N, w|_N)$. Since the worths of coalitions N_i , N_j and N are equal, we use *sub-games monotonicity* in both directions and obtain the following useful conditions on ψ :

$$\begin{aligned}\psi_p(N_i, w|_{N_i}) &= \psi_p(N_j, w|_{N_j}), \quad \forall p \in P = N_i \cap N_j, \\ \psi_i(N_i, w|_{N_i}) &= \psi_i(N, w|_N), \\ \psi_j(N_j, w|_{N_j}) &= \psi_j(N, w|_N).\end{aligned}$$

By *efficiency* we have the following condition on ψ :

$$\sum_{k \in N_i} \psi_k(N_i, w|_{N_i}) = w(N_i) = w(N_j) = \sum_{k \in N_j} \psi_k(N_j, w|_{N_j}).$$

Therefore we obtain that

$$\begin{aligned}\sum_{p \in P} \psi_p(N_j, w|_{N_j}) &= \sum_{p \in P} \psi_p(N_i, w|_{N_i}) \\ \Leftrightarrow \sum_{p \in P} \psi_p(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) &= \sum_{p \in P} \psi_p(N_i, w|_{N_i}) + \psi_i(N_i, w|_{N_i}) \\ \Leftrightarrow \sum_{p \in P} \psi_p(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) &= w(N_i) \\ \Leftrightarrow \sum_{p \in P} \psi_p(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) &= w(N_j) \\ \Leftrightarrow \sum_{p \in P} \psi_p(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) &= \sum_{p \in P} \psi_p(N_j, w|_{N_j}) + \psi_j(N_j, w|_{N_j}) \\ &\Leftrightarrow \psi_i(N_i, w|_{N_i}) = \psi_j(N_j, w|_{N_j}) \\ &\Leftrightarrow \psi_i(N, w|_N) = \psi_j(N, w|_N).\end{aligned}$$

The second and fourth equivalences come from efficiency. The third comes from the definition of w . The fifth and sixth come from sub-games monotonicity. We therefore obtain that $\psi_i(N, w|_N) = \psi_j(N, w|_N)$ which, by definition of w , is equivalent to $\psi_i(N, v) = \psi_j(N, v)$.

□

⁶If we consider a finite universe of players U , this construction is possible for any $N \subseteq U$ such that $n \leq \frac{u+1}{2}$.

Theorem 1

For every game $(N, v) \in \Gamma$, the sharing rule $\psi(N, v)$ satisfies **efficiency** and **sub-games monotonicity** if and only if $\psi = EG$.⁷

Proof. We know that EG satisfies *efficiency*. It is trivial that it satisfies *sub-games monotonicity*. Let us prove that, for any game $(N, v) \in \Gamma$, if a sharing rule ψ satisfies *efficiency* and *sub-games monotonicity* then it is the egalitarian value.

Let us consider a game (N, v) and a player $i \in N$. We can build a game (N_i, w_i) with N_i such that

$$\begin{aligned} N &\subsetneq N_i, \\ \exists T &\subsetneq N_i \text{ with } N \cap T = \{i\} \text{ and } n = t. \end{aligned}$$

We also put the following conditions on w_i :

$$\begin{aligned} w_i(P) &= v(P), \quad \forall P \subseteq N \\ w_i(T) &= v(N), \\ w_i(P) &= v(\{i\}), \quad \forall P \subsetneq T, \quad P \neq \{i\}. \end{aligned}$$

Such a construction is always possible since U is infinite.⁸ By definition we have $(w_i)_{|N}(S) = v(S)$ for all $S \subseteq N$ hence $\psi_i(N, (w_i)_{|N}) = \psi_i(N, v)$. Using *efficiency* and lemma 1 we have

$$\psi_i(T, (w_i)_{|T}) = \frac{(w_i)_{|T}(T)}{t} = \frac{v(N)}{n}$$

since all players $j \in T$ are equals in $(w_i)_{|T}$. As we have $N, T \subseteq N_i$, $n = t$ and $w_i(T) = w_i(N)$ using sub-games monotonicity we obtain that

$$\psi_i(T, (w_i)_{|T}) \geq \psi_i(N, (w_i)_{|N}) \text{ and } \psi_i(N, (w_i)_{|N}) \geq \psi_i(T, (w_i)_{|T}),$$

⁷If we consider a finite universe of players U , this result holds for any $N \subseteq U$ such that $n \leq \frac{u+1}{2}$.

⁸If we consider a finite universe of players U , this construction is possible for any $N \subseteq U$ such that $n \leq \frac{u+1}{2}$.

which means that

$$\psi_i(N, (w_i)_{|_N}) = \psi_i(T, (w_i)_{|_T}) = \frac{v(N)}{n} = \psi_i(N, v).$$

Since U is an infinite set it is possible to build a game (N_i, w_i) that satisfies our above conditions for every $i \in N$. Hence we have that $\psi_i(N, v) = \frac{v(N)}{n}$, $\forall i \in N$, which concludes our proof. □

We now prove that our axioms are logically independent.

Dropping efficiency

Consider the sharing rule $f^1(N, v) = \alpha EG(N, v)$ with $\alpha \neq 1$. It obviously satisfies *sub-games monotonicity* but not *efficiency*.

Dropping sub-games monotonicity

Consider the Shapley value. It satisfies efficiency. Example 1 shows it does not satisfy sub-games monotonicity.

We build the axiom of *sub-games monotonicity* on the following idea of coalitional stability: a coalition can only form if each player consents to its formation. This conception is in line with the model 1 of stability presented in [Hart and Kurz \(1983\)](#). Additionally, this conception can also be interpreted as a requirement for a deviation. Therefore, in this sense, coalitional stability can be thought as the inability for players to form another coalition than the one they are currently in. This gives us ground on which we define a weaker version of sub-games monotonicity :

Weak sub-games monotonicity. *For each game $(N, v) \in \Gamma$ with $n \geq 3$ and each couple of coalitions $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$, if $v(S) \geq v(T)$ it holds that $\exists i \in S \cap T$ such that $\psi_i(S, v_{|_S}) \geq \psi_i(T, v_{|_T})$.*

However, lemma 1 does not hold with this weaker version of sub-games

monotonicity as we see with the following counter-example.

Counter-example. Let us consider the sharing rule f^2 such that

$$f_i^2(N, v) = \begin{cases} v(N) & \text{if } i = \min(N), \\ 0 & \text{else,} \end{cases}$$

where $\min(N)$ denotes the first player in the lexicographical order of the set N . This sharing rule obviously satisfies *efficiency* and violates *equal treatment of equals*. We show it satisfies the *weak sub-games monotonicity* axiom.

Consider a game (N, v) and two subcoalitions $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$. Let us denote $i = \min(S)$. We first consider the case where $v(S) > v(T)$. We have

$$f_i^2(S, v|_S) = v(S) > f_i^2(T, v|_T) = \begin{cases} v(T) & \text{if } i = \min(T), \\ 0 & \text{else.} \end{cases}$$

Hence there exists a player $k \in S \cap T$ such that $f_k^2(S, v|_S) \geq f_k^2(T, v|_T)$.

We now consider the case where $v(S) = v(T)$. We have

$$f_i^2(S, v|_S) = v(S) \geq f_i^2(T, v|_T) = \begin{cases} v(T) = v(S) & \text{if } i = \min(T), \\ 0 & \text{else.} \end{cases}$$

Hence there exists a player $k \in S \cap T$ such that $f_k^2(S, v|_S) \geq f_k^2(T, v|_T)$. Similarly if we consider a player $j \in S \cap T$ such that $j \neq i$ we have

$$f_j^2(S, v|_S) = 0 \leq f_j^2(T, v|_T) = \begin{cases} v(T) & \text{if } j = \min(T), \\ 0 & \text{else.} \end{cases}$$

Therefore there also exists a player $k \in S \cap T$ such that $f_k^2(S, v|_S) \leq f_k^2(T, v|_T)$. □

We can infer from this counter-example that lemma 1 and theorem 1 do not hold when we replace *sub-games monotonicity* with its weaker variant. However

we can obtain the following result.

Theorem 2

*For every game $(N, v) \in \Gamma$, the sharing rule $\psi(N, v)$ satisfies **efficiency, equal treatment of equals and weak sub-games monotonicity** if and only if $\psi = EG$.⁹*

Proof. In the proof of theorem 1 we used a construction where $S \cap T$ is a singleton. Therefore the proof for this theorem is identical except we use the equal treatment of equals axiom where lemma 1 was previously used. □

Note that the three axioms are independent : Sh satisfies *equal treatment of equals, efficiency* but not *weak sub-games monotonicity* as shown in example 2; f^1 satisfies *equal treatment of equals and weak sub-games monotonicity* but not *efficiency*; finally, as shown with the above counter-example, f^2 satisfies *efficiency and weak sub-games monotonicity* but not *equal treatment of equals*.¹⁰

We now look at a stronger variant of *sub-games monotonicity*. On top of the requirement of *sub-games monotonicity* we consider that each player i from a coalition S can be swapped for any player p in $N \setminus S$. We then ask that every players in S earn a payoff at least as high as any player p can earn in any other coalition T of the same size, provided that the worth of T is lower than the worth of S . Formally,

⁹If we consider a finite universe of players U , this result holds for any $N \subseteq U$ such that $n \leq \frac{u+1}{2}$.

¹⁰It is to be noted that theorem 2 still holds when *weak sub-games monotonicity* is defined with strict inequality. The proof is quite similar to the proof of theorem 1. For every game (N, v) and every player $i \in N$, build two games (N_i, w_1) and (N_i, w_2) with N_i being defined as in the proof of theorem 1. Build w_1 and w_2 as we did w_i in the aforementioned proof with the difference that $w_1(T) = v(N) - \epsilon$ and $w_2(T) = v(N) + \epsilon$, $\epsilon \in \mathbb{R}$. Using *efficiency* and *equal treatment of equals* we obtain an upper bound $(\frac{v(N)+\epsilon}{n})$ and a lower bound $(\frac{v(N)-\epsilon}{n})$ for $\psi_i(N, v)$. Since ϵ can be arbitrarily small we conclude the proof. The axioms are also independent as f_2 still satisfies *weak sub-games monotonicity* with strict inequalities.

Strong sub-games monotonicity. For each game $(N, v) \in \Gamma$ with $n \geq 3$ and each couple of coalitions $S, T \subseteq N$ with $s = t$ and $v(S) \geq v(T)$ it holds that $\psi_i(S, v|_S) \geq \psi_p(T, v|_T)$, $\forall i \in S, \forall p \in (N \setminus S) \cup i, |T \cap \{p, i\}| = 1$.

Theorem 3

For every game $(N, v) \in \Gamma$, the sharing rule $\psi(N, v)$ satisfies **efficiency** and **strong sub-games monotonicity** if and only if $\psi = EG$.¹¹

Proof. We first prove that *EG* satisfies *strong sub-games monotonicity*. Let (N, v) be a TU-game with $n \geq 3$. Take a coalition $S \subseteq N$, we have $EG_i(S, v|_S) = \frac{v(S)}{s}, \forall i \in S$. Consider now any coalition $T \subseteq N$ such that $s = t, v(S) \geq v(T)$. We have $EG_j(T, v|_T) = \frac{v(T)}{t} = \frac{v(T)}{s} \leq \frac{v(S)}{s}, \forall j \in T$.

Now for the uniqueness part. It is enough to show that *strong sub-games monotonicity* implies *sub-games monotonicity*. Let (N, v) be a TU-game with $n \geq 3$ and consider any couple of coalition $S, T \subseteq N$, with $s = t$, and $v(S) \geq v(T)$. By *strong sub-games monotonicity* we must have that

$$\psi_i(S, v|_S) \geq \psi_p(T, v|_T), \forall i \in S, \forall p \in (N \setminus S) \cup i, |T \cap \{p, i\}| = 1.$$

We necessarily have $S \cap T \neq \emptyset$. In the particular cases where $p = i$ then we have

$$\psi_i(S, v|_S) \geq \psi_i(T, v|_T), \forall i \in S \cap T.$$

□

¹¹If we consider a finite universe of players U , this result holds for any $N \subseteq U$ such that $n \leq \frac{u+1}{2}$.

3. Games with types

We introduce a new model of cooperative games in which players are of a given type.

Consider an entity (a firm for instance) is setting up a project that asks for specific types of players. It potentially has several players (candidates, employees, other firms) of each type. Each player is of a given type but several players are of the same type. Once the team is put together, the player that are parts of the team will work on the project which implies working all together as well as in subgroups and alone.

Formally, we consider k types with $1 < k < n$. Each player $i \in N$ is of only one type and we denote by $K_\alpha \subsetneq N$ the set of players of type $\alpha \in \{1, \dots, k\}$. We write $K = \{K_1, K_2, \dots, K_k\}$ the set of those sets. We write K^i the set of players of the same type as player i . Noting $\mathcal{P}(N)$ the partitions of the player set N , the set $K \in \mathcal{P}(N)$ can be understood as a coalition structure as introduced by [Aumann and Dreze \(1974\)](#). We have $\bigcup_{1 \leq \alpha \leq k} K_\alpha = N$ and $K_{\alpha_1} \cap K_{\alpha_2} = \emptyset$ for any $\alpha_1, \alpha_2 \in \{1, \dots, k\}$ and $\alpha_1 \neq \alpha_2$.

We name *full coalition* a coalition where each type is present. We denote by $F = \prod_{1 \leq \alpha \leq k} K_\alpha$ the set of full coalitions, defined by the cartesian products of the sets that partition players into types. Therefore, the size of each full coalition is k . We also note $F^i = \{F' \in F \mid i \in F'\}$ the set of full coalitions to which player i belongs. We assume that sub-coalitions of a full coalitions are still feasible¹² but that coalitions with excess players are not. Hence the set of feasible coalitions $\mathcal{F}_{N,K}$ is the union of the subsets of full coalitions.

We consider that a value is generated from the cooperation of players. This production is modelled by a characteristic function v defined over 2^N . Hence, we define a game with types as a triplet (N, v, K) with $N \subseteq U$, $(N, v) \in \Gamma$ and

¹²A feasible coalition is a coalition that can be formed by players.

$K \in \mathcal{P}(N)$. We denote the set of those games by Γ_{type} .

We provide an example based on example 1 from section 2.

Example 3 (*Team project*)

Consider the following situation: a business trusts one of its managers to take on a new project that needs two other specialists. She is given freedom to choose other employees of the necessary types. The other employees are free to reject the proposition. Once the project is done, the team will get a bonus proportional to the estimated value of the work produced. This raises the problem of finding a sharing rule for this bonus that gives the manager the incentives to put together the team that produce the most valuable work, and for the potential teammates to accept. Formally we have $N = \{1, 2, 3, 4, 5\}$ and $K = \{\{1\}, \{2, 3\}, \{4, 5\}\}$. The full coalitions are $F = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}\}$. The set of feasible coalition is given by

$$\mathcal{F}_{N,K} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}\}.$$

Imagine that 2 has poor synergy with both 4 and 5 while 3 has a really good synergy with 5 (only amplified under 1 management) but can't stand working with 4. However, as a manager, 1 is able to be at its best when bridging the divide between 4 and 3. The following worths for the feasible coalitions express this situation. The worths express the value of intermediary works as well as the end product. We have $v(\{i\}) = 0$ for all $i \in N$ and ¹³

S	12	13	14	15	24	25	34	35	124	125	134	135
$v(S)$	0.5	0	1	1	0.5	0.75	0	2	2	2	3	4

■

¹³Note that v is defined for any $S \subseteq N$. We only focus here on the feasible coalitions so as not to take up too much space.

This example illustrates the three issues that naturally arise in this model: the formation of one of the full coalitions, the allocation of its worth, and the influence of this allocation on the stability of the formation process of the selected full coalition. To address these issues, we define a solution on Γ_{type} as a pair $\phi = (\mu, \psi)$. The function μ is a mechanism that selects for each $(N, v, K) \in \Gamma_{type}$ one of its full coalitions $\mu(N, v, K)$. The function ψ is a sharing rule on Γ . For a given game with types $(N, v, K) \in \Gamma_{type}$ a solution is $\phi(N, v, K) = (\mu(N, v, K), \psi(\mu(N, v, K), v_{|\mu(N, v, K)}))$. Such a solution is therefore two-dimensional. We shall then put requirements on both dimensions separately as well as a requirement on their interaction.

One desirable and reasonable requirement for a mechanism μ would be that it selects an optimal full coalition. A full coalition $F' \in F$ is optimal if and only if $v(F') = \max_{F'' \in F} v(F'')$. We let $F_{opt} = \{F' \in F \mid v(F') = \max_{F'' \in F} v(F'')\}$ be the set of optimal full coalitions. Hence, we ask that the mechanism μ selects a coalition that is optimal.¹⁴ This gives us a first axiom:

Optimality. For every games with types $(N, v, K) \in \Gamma_{type}$ it holds that $\mu(N, v, K) \in F_{opt}$.

Next we put a requirement on the sharing rule ψ on Γ . We require that ψ satisfies the standard axiom of **efficiency** (as defined in section 2). Observe that even though we consider a model of games with types, ψ is a sharing rule for TU-games. Hence it is sufficient to require that ψ is efficient in order to ensure that the worth of the full coalition is completely allocated amongst its players.

Finally, we introduce an axiom that concerns the interaction between μ and ψ . Given a solution $\phi = (\mu, \psi)$, can we make sure that the coalition $\mu(N, v, K)$ is stable? That is to say, given ψ , can we make sure that no players in $\mu(N, v, K)$ would be better off associating with other players? We consider the payoff a

¹⁴Note that there can be multiple optimal full coalitions. We only ask that μ selects one of these. Hence it is possible that μ is not anonymous.

player can obtain with another coalition as a player's outside options. Hence the following question : does the solution $\phi = (\mu, \psi)$ is such that no player that belongs to $\mu(N, v, K)$ has better outside options ?¹⁵ Formally:

No outside options. *For every game with types $(N, v, K) \in \Gamma_{type}$ it holds that $\phi = (\mu, \psi)$ is such that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F', v_{|F'}), \forall F' \in F^i, \forall i \in \mu(N, v, K)$.*

Remark. From example 1 and 3 it is easy to see that if ψ is chosen to be the Shapley value then ϕ does not satisfy *no outside options*.

The axiom of *no outside options* is defined for a solution ϕ defined on Γ_{type} which itself is entirely composed of a solution ψ defined on Γ and of a coalition selector μ . Hence the requirement that is put on ϕ by the axiom of *no outside options* necessarily interacts with the requirement put on μ by *optimality* and the requirement put on ψ by *efficiency*. The next two results explore how these three axioms interact. In particular, the next proposition explore how, when requiring *optimality*, the *no outside options* axiom can be “translated” to a specific requirements on ψ .

Proposition 1

*Let $(N, v, K) \in \Gamma_{type}$ be a game with type and let $\phi = (\mu, \psi)$ be solution for such games. Requiring that ϕ satisfies **no outside options** for every μ that satisfies **optimality** is equivalent to requiring that ψ satisfies **sub-games monotonicity**.*

Proof. We first prove that if μ satisfies *optimality* and ψ satisfies *sub-games monotonicity* then ϕ satisfies *no outside options*.

¹⁵The concept of outside options considered here is in line with the one used by Casajus (2009) in the context of communication graph games.

Let us consider a solution $\phi = (\mu, \psi)$. If μ satisfies *optimality* then $\mu(N, v, K) \in F_{opt}$ which implies that $v(\mu(N, v, K)) \geq v(F')$ for all $F' \in F$. By definition we have that $|\mu(N, v, K)| = |F'|$, for all $F' \in F$. Using *sub-games monotonicity* we know then that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F', v_{|F'})$ for all F' such that $\mu(N, v, K) \cap F' \neq \emptyset, \forall i \in \mu(N, v, K) \cap F'$. By definition, if $i \in \mu(N, v, K) \cap F'$ then $F' \in F^i$. Hence $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F', v_{|F'})$ for all $F' \in F^i$, for all $i \in \mu(N, v, K)$. This coincides with the requirement for *no outside options*.

We now prove that if ϕ satisfies **no outside options** for any μ that satisfies **optimality** then ψ satisfies **sub-games monotonicity**.

Let (N, v) be a TU-game. It is sufficient to show that for any $S, T \subseteq N$ with $s = t, S \cap T \neq \emptyset$ and $v(S) \geq v(T)$ there exists a game with types (N, w, K) such that $\mu(N, w, K) = S$ and $T \in F$.

Let (N, w, K) be a game with types. We build w such that

- $w(R) = v(R), \forall R \subseteq S$ and $\forall R \subseteq T$;
- $w(R) < v(S), \forall R \subseteq N, r = s, R \neq S, R \neq T$.

Additionally we consider a partition K of N such that $k = s$. Assume that every $i \in S$ is in a different element of K and that every $j \in T$ is also in a different element of K . This ensures that S and T are full coalitions. The *no outside options* axiom impose the following conditions:

$$\left\{ \begin{array}{l} \psi_i(\mu(N, w, K), w_{|\mu(N, w, K)}) \geq \psi_i(F', w_{|F'}), \quad \forall F' \in F, F' \cap \mu(N, w, K) \neq \emptyset, \\ \forall i \in F' \cap \mu(N, w, K). \end{array} \right.$$

We first assume that $v(S) > v(T)$. Using optimality on μ we obtain

$$\begin{aligned} &\Leftrightarrow \left\{ \begin{array}{l} \psi_i(S, w_{|S}) \geq \psi_i(F', w_{|F'}), \quad \forall F' \in F, F' \cap S \neq \emptyset, \\ \forall i \in F' \cap S. \end{array} \right. \\ &\Rightarrow \left\{ \begin{array}{l} \psi_i(S, v_{|S}) \geq \psi_i(T, v_{|T}), \quad \forall i \in T \cap S. \end{array} \right. \end{aligned}$$

The last implication comes from the facts that $T \in F$, $w|_S = v|_S$ and $w|_T = v|_T$. This last condition coincides with the one for *sub-games monotonicity*.

We now assume that $v(S) = v(T)$. We want the solution ϕ to satisfies no outside options for any possible μ that is optimal. If $v(S) = v(T)$ then, in the game (N, w, K) we have $F_{opt} = \{S, T\}$. Since an optimal coalition selector μ can choose either of these two coalitions, the conditions under which ϕ satisfies no outside option, for any μ that is optimal become :

$$\Leftrightarrow \begin{cases} \psi_i(S, w|_S) \geq \psi_i(F', w|_{F'}), & \forall F' \in F, F' \cap S \neq \emptyset, \\ & \forall i \in F' \cap S; \\ \psi_i(S, w|_S) \leq \psi_i(F', w|_{F'}), & \forall F' \in F, F' \cap S \neq \emptyset, \\ & \forall i \in F' \cap S. \end{cases}$$

$$\Rightarrow \left\{ \psi_i(S, v|_S) = \psi_i(T, v|_T), \quad \forall i \in T \cap S. \right.$$

This condition coincides with the condition of sub-games monotonicity. □

Theorem 4

For every game with type $(N, v, K) \in \Gamma_{type}$ such that $k \leq \frac{n+1}{2}$. Consider the solution $\phi = (\mu, \psi)$ with ψ satisfying **efficiency**. The solution ϕ satisfies **no outside options** for every μ that satisfies **optimality** if and only if $\psi = EG$.

Proof. We first prove the existence part. We consider a solution $\phi = (\mu, EG)$ with μ satisfying *optimality*. We know that *EG* satisfies *efficiency* and *sub-games monotonicity*. From proposition 1 we conclude that $\phi = (\mu, EG)$ satisfies *no outside options*.

Now the uniqueness part. We consider a solution $\phi = (\mu, \psi)$ with ψ satisfying *efficiency*. Since ϕ satisfies *no outside options* for any μ satisfying *optimality*, from proposition 1 we know that ψ satisfies *sub-games monotonicity*. Given the game (N, v, K) , every full coalition is of size k . Since $k \leq \frac{n+1}{2}$ we can use theorem 1 to conclude that $\psi = EG$.

□

As we did in section 2 for *sub-games monotonicity*, we can design weaker or stronger variants of the *no outside options* axiom. The weaker variant relies on the same conception of deviation and coalitional stability that we presented in section 2.

No outside options (weak). *For every game with types $(N, v, K) \in \Gamma_{type}$, it holds that the solution $\phi(N, v, K) = (\mu, \psi)$ is such that $\exists i \in \mu(N, v, K) \cap F'$ such that*

$$\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F', v_{|F'}), \quad \forall F' \in F.$$

An alternative coalition F' can form only if no member of $\mu(N, v, K) \cap F'$ object to its formation. We obtain a parallel result to proposition 1.

Proposition 2

*Let $(N, v, K) \in \Gamma_{type}$ be a game with type and let $\phi = (\mu, \psi)$ be solution for such games. Requiring that ϕ satisfies **no outside options (weak)** for every μ that satisfies **optimality** is equivalent to requiring that ψ satisfies **weak sub-games monotonicity**.*

Proof. We first prove that if μ satisfies *optimality* and ψ satisfies *weak sub-games monotonicity* then ϕ satisfies *no outside options (weak)*.

Let $\phi = (\mu, \psi)$ be a solution for games with types. Let $(N, v, K) \in \Gamma_{type}$ be such a game. Since μ satisfies *optimality* then $v(\mu(N, v, K)) \geq v(F')$ for all $F' \in F$. In particular we have $v(\mu(N, v, K)) \geq v(F')$ for all $F' \in F$ such that $F' \cap \mu(N, v, K) \neq \emptyset$. Since ψ satisfies *weak sub-games monotonicity* we obtain that, $\forall F' \in F$ with $\mu(N, v, K) \cap F' \neq \emptyset$, $\exists i \in \mu(N, v, K) \cap F'$ such that

$$\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F', v_{|F'}).$$

Which coincides with the *no outside options (weak)* axiom.

We now prove that if ϕ satisfies *no outside options (weak)* for any μ that satisfies

optimality then ψ satisfies *weak sub-games monotonicity*. Let (N, v) be a TU-game. It is sufficient to show that for any $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$ and $v(S) \geq v(T)$ there exists a game with types (N, w, K) such that $\mu(N, w, K) = S$ and $T \in F$.

Let (N, w, K) be a game with types. We build w such that

- $w(R) = v(R)$, $\forall R \subseteq S$ and $\forall R \subseteq T$;
- $w(R) < v(S)$, $\forall R \subseteq N$, $r = s$, $R \neq S$, $R \neq T$.

Additionally we consider a partition K of N such that $k = s$. Assume that every $i \in S$ is in a different element of K and that every $j \in T$ is also in a different element of K . This ensures that S and T are full coalitions. The *no outside options (weak)* axiom impose the following condition:

$$\left\{ \begin{array}{l} \forall F' \in F, F' \cap \mu(N, w, K) \neq \emptyset, \exists i \in F' \cap \mu(N, w, K) \text{ such that :} \\ \psi_i(\mu(N, w, K), w_{|\mu(N, w, K)}) \geq \psi_i(F', w_{|F'}). \end{array} \right.$$

We first assume that $v(S) > v(T)$. Using *optimality* on μ we obtain

$$\begin{aligned} &\Leftrightarrow \left\{ \begin{array}{l} \forall F' \in F, F' \cap S \neq \emptyset, \exists i \in F' \cap S \text{ such that :} \\ \psi_i(S, w_{|S}) \geq \psi_i(F', w_{|F'}), \end{array} \right. \\ &\Rightarrow \left\{ \begin{array}{l} \exists i \in S \cap T \text{ such that } \psi_i(S, v_{|S}) \geq \psi_i(T, v_{|T}). \end{array} \right. \end{aligned}$$

The last implication comes from the facts that $T \in F$, $w_{|S} = v_{|S}$ and $w_{|T} = v_{|T}$. This last condition coincides with the one for *weak sub-games monotonicity*.

We now assume that $v(S) = v(T)$. We want the solution ϕ to satisfies *weak no outside options* for any possible μ that is optimal. If $v(S) = v(T)$ then, in the game (N, w, K) we have $F_{opt} = \{S, T\}$. Since an optimal coalition selector μ can choose either of these two coalitions, the condition under which ϕ satisfies

no outside option (weak), for any μ that is optimal becomes :

$$\Leftrightarrow \begin{cases} \forall F' \in F, F' \cap S \neq \emptyset, \exists i \in F' \cap S \text{ such that :} \\ \psi_i(S, w_{|S}) \geq \psi_i(F', w_{|F'}), \\ \forall F' \in F, F' \cap S \neq \emptyset, \exists i \in F' \cap S \text{ such that :} \\ \psi_i(S, w_{|S}) \leq \psi_i(F', w_{|F'}). \end{cases}$$

$$\Rightarrow \begin{cases} \exists i \in S \cap T \text{ such that } \psi_i(S, w_{|S}) \geq \psi_i(T, w_{|T}), \\ \exists j \in S \cap T \text{ such that } \psi_j(S, w_{|S}) \leq \psi_j(T, w_{|T}). \end{cases}$$

This condition coincides with the condition of *weak sub-games monotonicity* when $v(S) = v(T)$.

□

Theorem 5

For every game with types $(N, v, K) \in \Gamma_{type}$ such that $k \leq \frac{n+1}{2}$. Consider the solution $\phi = (\mu, \psi)$ with ψ satisfying **efficiency** and **equal treatment of equals**. The solution ϕ satisfies **no outside options (weak)** for every μ that satisfies **optimality** if and only if $\psi = EG$.

Proof. We first prove the existence part. We consider a solution $\phi = (\mu, EG)$ with μ satisfying *optimality*. We know that *EG* satisfies *efficiency*, *equal treatment of equals* and *weak sub-games monotonicity*. From proposition 2 we conclude that $\phi = (\mu, EG)$ satisfies *no outside options (weak)*.

Now the uniqueness part. We consider a solution $\phi = (\mu, \psi)$ with ψ satisfying *efficiency* and *equal treatment of equals*. Since ϕ satisfies *no outside options (weak)* for every μ satisfying *optimality*, from proposition 2 we know that ψ satisfies *weak sub-games monotonicity*. Given the game (N, v, K) , every full coalition is of size k . Since $k \leq \frac{n+1}{2}$ we can use theorem 2 to conclude that $\psi = EG$.

□

We now consider a more restrictive version of the *no outside options* axiom. We require that the payoff obtained by each player within the chosen full coal-

tion $\mu(N, v, K)$ should be at least as large as the payoff that any other player of the same type can obtain in any other full coalition. Formally:

No outside options (strong). For every game with types $(N, v, K) \in \Gamma_{type}$ it holds that $\phi = (\mu, \psi)$ is such that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_j(F', v_{|F'})$, $\forall i \in \mu(N, v, K), \forall j \in K^i, \forall F' \in F^j$.

In this stronger version we compare the payoff of a player i of the selected full coalition $\mu(N, v, K)$ with every payoff of every other players being of the same type as i in every full coalitions. It is clear that this axiom is a stronger version of *no outside options* and that *no outside options (strong)* implies *no outside options*. We now prove the following result.

Proposition 3

Let $(N, v, K) \in \Gamma_{type}$ be a game with types and let $\phi = (\mu, \psi)$ be solution for such games. Requiring that ϕ satisfies **no outside options (strong)** for every μ that satisfies **optimality** is equivalent to requiring that ψ satisfies **strong sub-games monotonicity**.

Proof. We first prove that if μ satisfies *optimality* and ψ satisfies *strong sub-games monotonicity* then ϕ satisfies *no outside options (strong)*.

Let $\phi = (\mu, \psi)$ be a solution for games with types. Let $(N, v, K) \in \Gamma_{type}$ be such a game. Since μ satisfies *optimality* then $v(\mu(N, v, K)) \geq v(F')$ for all $F' \in F$.

Since ψ satisfies *strong sub-games monotonicity* we have

$$\left\{ \begin{array}{l} \forall i \in \mu(N, v, K), \\ \psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_p(F', v_{|F'}), \quad \forall p \in (N \setminus \mu(N, v, K)) \cup \{i\}, \\ \forall F' \in F, |F' \cap \{p, i\}| = 1. \end{array} \right.$$

As $i \in \mu(N, v, K)$, any $j \in K^i$ also belongs to $(N \setminus \mu(N, v, K)) \cup \{i\}$. In addition,

if $j \in F'$ then $F' \in F^j$. Hence the above conditions implies that

$$\left\{ \begin{array}{l} \forall i \in \mu(N, v, K), \\ \psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_j(F', v_{|F'}), \quad \forall j \in K^i, \\ \forall F' \in F^j. \end{array} \right.$$

Which coincides with *no outside options (strong)*.

We now prove that if ϕ satisfies *no outside options (strong)* for any μ that satisfies *optimality* then ψ satisfies *strong sub-games monotonicity*. Let (N, v) be a TU-game. We consider a coalition $S \subseteq N$ and a player $i \in S$. We also consider a player $p \in (N \setminus S) \cup \{i\}$ and a coalition $T \subseteq N$ such that $s = t$, $v(S) \geq v(T)$ and $|T \cap \{i, p\}| = 1$. Observe that if $p = i$ we have the same setup than in the proof of theorem 4. Therefore we assume in the following that $p \neq i$. Thus $p \in T$.

We build a game with types (N, w, K) with $w(R) = v(R)$, $\forall R \subseteq S$, $\forall R \subseteq T$ and $w(R) < v(R)$, $\forall R \neq S$, $\forall R \neq T$ with $r = s$. We build the partition K such that $k = s$ and every $j \in S$ is of a different type. We assume that every $j \in T$ is of a different type as well. In particular, we build K such that $p \in K^i$. Therefore $p \in (N \setminus S) \cup \{i\}$, $S, T \in F$ and $T \in F^p$. Since ϕ satisfies *no outside options (strong)* we have that

$$\psi_j(\mu(N, w, K), v_{|\mu(N, w, K)}) \geq \psi_l(F', w_{|F'}), \quad \forall j \in \mu(N, w, K), \quad \forall l \in K^j, \quad \forall F' \in F^l.$$

We first assume that $v(S) > v(T)$. The above conditions then become

$$\psi_j(S, w_{|S}) \geq \psi_l(F', w_{|F'}), \quad \forall j \in S, \quad \forall l \in K^i, \quad \forall F' \in F^l.$$

In particular we have

$$\begin{aligned} \psi_i(S, w_{|S}) &\geq \psi_p(T, w_{|T}) \\ \Leftrightarrow \psi_i(S, v_{|S}) &\geq \psi_p(T, v_{|T}). \end{aligned}$$

We showed we can build a game with types (N, w, K) giving us this condition for any $i \in S$, any $p \in (N \setminus S) \cup \{i\}$ and any $T \subseteq N$ such that $s = t$, $v(S) \geq v(T)$

and $|T \cap \{i, p\}| = 1$. This condition therefore coincide with *strong sub-game monotonicity*.

We now assume that $v(S) = v(T)$. We want the solution ϕ to satisfies *no outside options (strong)* for any possible μ that is optimal. If $v(S) = v(T)$ then, in the game (N, w, K) we have $F_{opt} = \{S, T\}$. Note that, if we have a coalition T with $s = t$ such that, for a given $i \in S$ and $p \in (N \setminus S) \cup \{i\}$ we have $|T \cap \{i, p\}| = 1$ we can “reverse” this construction. Take T as given, we have $p \in T$ and $i \in (N \setminus T) \cup \{p\}$. We have necessarily that $|S \cap \{i, p\}| = 1$.

Therefore, since an optimal coalition selector μ can choose either S or T , the condition under which ϕ satisfies *no outside option (strong)*, for any μ that is optimal becomes :

$$\psi_i(S, v_{|S}) \geq \psi_p(T, v_{|T}) \text{ and } \psi_i(S, v_{|S}) \leq \psi_p(T, v_{|T}).$$

This condition coincides with the condition of *strong sub-games monotonicity* when $v(S) = v(T)$.

□

Theorem 6

For every game with type $(N, v, K) \in \Gamma_{type}$ such that $k \leq \frac{n+1}{2}$. Consider the solution $\phi = (\mu, \psi)$ with ψ satisfying **efficiency**. The solution ϕ satisfies **no outside options (strong)** for every μ that satisfies **optimality** if and only if $\psi = EG$.

Proof. We first prove the existence part. Consider a solution $\phi = (\mu, EG)$ with μ satisfying *optimality*. We know that *EG* satisfies *efficiency*. We have

$$\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) = \frac{v_{|\mu(N, v, K)}(\mu(N, v, K))}{k}, \quad \forall i \in \mu(N, v, K),$$

as well as

$$\psi_j(F', v_{|F'}) = \frac{v_{|F'}(F')}{k}, \quad \forall F' \in F^j, \quad \forall j \in N.$$

Since μ satisfies *optimality* we have $v_{|\mu(N,v,K)}(\mu(N,v,K)) \geq v_{|F'}(F') \forall F' \in F$. Hence, $\phi = (\mu, ED)$ satisfies *no outside options (strong)*.

Now the uniqueness part. We consider a solution $\phi = (\mu, \psi)$ with μ satisfying *optimality* and ψ satisfying *efficiency*. Since ϕ satisfies *no outside options (strong)*, it satisfies *no outside options*. From proposition 1 we know that ψ satisfies *sub-games monotonicity*. Given the game (N, v, K) , every full coalition is of size k . Since $k \leq \frac{n+1}{2}$ we can use theorem 1 to conclude that $\psi = EG$.

□

4. Concluding remarks

In this paper we developed a model of cooperative environments in which different types of players are needed. This model is built on the canonical model of cooperative TU-games. We showed that, if the number of players is sufficiently larger than the number of types, an egalitarian approach when sharing the outcome of cooperation is the only efficient approach that always provide incentive for the best coalition to form. We showed that this result is still true when a stronger definition of stability is considered but require the additional axiom of *equal treatment of equals* when a weaker variant is considered.

In our approach, the full domain of cooperative games is considered. Example 1 and 2 show that the solution $\phi = (\mu, Sh)$ does not satisfy no outside options even for convex games. On specific classes of games, it would be interesting to determine whether other sharing rules than the egalitarian value are compatible with both the axioms of *sub-games monotonicity* and *efficiency*.

In addition, it is well known that the payoff vector given by the egalitarian value often lies outside the core, even for convex games. Consequently the selected full coalition can offer no outside options but not be stable in the sense of the core. As such our result could also be framed as an impossibility result: there are no solution $\phi = (\mu, \psi)$ with ϕ satisfying *no outside options* for every μ satisfying *optimality* such that $\psi(\mu(N, v, K), v_{|\mu(N,v,K)})$ lies in the core of

$(\mu(N, v, K), v|_{\mu(N, v, K)})$. However, the fact that ψ might not be in the core can possibly be irrelevant if we consider situations where the cooperation is based on free association but, once established, is subject to binding agreements.

Furthermore, our result can help understand the pervasiveness of fixed remunerations in organizations, which is still a challenge for incentives theory to this day (Ménard, 2012). Consider a firm where employees are on a fixed wage determined by their position. For instance, accountants earn x , designers earn y and so on. This corresponds in our model to a solution where the payoff ψ_i of a player i is determined by its type, independently of the total worth produced. Such a solution would satisfy *sub-games monotonicity*. Indeed, since every payoffs is a fixed amount determined exogenously there are no differences depending on the sub-coalition considered. A player payoff is always the same regardless of the worth produced. However it is not *efficient*. In a capitalist firm, the difference between the sum of the wages and the value produced is the residual surplus that goes to the owner of the firm's capital, as losses or profit. The capitalist firm, since it is not efficient, can therefore be sub-games monotonic while having differentiated wages.

Finally, on the basis of the main result of this paper, we argue that egalitarianism as a solution can provide some form of incentives towards optimality in cooperative situations that satisfies the following conditions:

- cooperation is based on free association;
- the numbers of players needed is sufficiently smaller than the numbers of candidates.

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