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“One person, one n^{th} !”:
The incentive effects of egalitarianism in cooperative
environments¹

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Abstract

This article introduces a new axiom of intersecting population monotonicity for TU-games as well as a new model. The axiom, alongside efficiency, characterizes the egalitarian value. The new model addresses situations where a number of roles have to be performed by players. Each player have a specific role and coalitions are feasible only if it contains at most one player for each role. We use the new characterization of the egalitarian value for TU-games to obtain the following result in our class of problem: the egalitarian value is the only sharing rule that ensures that each player of the most productive group is better off joining this most productive group. Thus, we characterize the egalitarian value without fairness requirement and show that, for this new class of problem, egalitarianism can provide incentives towards optimal cooperation.

Keywords: Cooperative game theory, shapley value, equal division, egalitarian value, role structure, incentives

1. Introduction

In the present article we develop a model of cooperative games in which a number of roles have to be filled by players. Each role can be perform by a precise set of

¹I would like to thank Sylvain Béal who was always available to provide insightful feedbacks and comments at various stages of this paper. I also thank Stéphane Gonzalez for seeing some merits in the original draft, which gave me the confidence to pursue this idea.

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players, and players can only perform one role. The role structure restricts the feasibility of coalitions: a coalition is feasible if it contains at most one player performing each role. This means we have situations in which the grand coalition cannot form. Such a situation can arise when members of an organization have to form a subgroup to tackle a specific project (see application 2). Several groups having to choose a representative to participate in a committee can also be a situation covered by our model. A central planner having to manage the use of a public resource between different actors can be a third (see application 3). The problem raised by our model is thus different than the usual problem of cooperative games (sharing the worth of the grand coalition).

A solution ϕ to this model consists of two mechanisms: a function μ that selects a group of players specifying, for each role, a single player being able to perform the role, and a sharing rule ψ that shares the worth produced by the group amongst its members. We are interested in solution that fosters the emergence of the most productive group that fills all role. To this end, we propose an axiom of *optimality* (the solution selects the best group) and an axiom of *no outside options* (no member of the selected group can be better off in another group) that apply to the solution ϕ . Together, these axioms imply requirements on the allocation problem for the chosen group. We introduce an axiom of *intersecting population monotonicity* for the sharing rule ψ . This axiom asks that no member of a population can be worse off by choosing to associate with a group of a given size with which it has the best synergy. We show that if μ satisfies *optimality*, then the axiom of *no outside option* on ϕ and of *intersecting population monotonicity* on ψ are equivalent. As an intermediary result, we offer a characterization of the egalitarian value (which shares the total worth equally between all players) that relies on the *intersecting population monotonicity* axiom alongside the standard axiom of *efficiency*. As such we characterize the egalitarian principle without any fairness criteria.

Formally, our model is related to the Aumann-Drèze model (Aumann and Dreze, 1974) as it also involves coalition structure. In a coalition structure, players are part of pre-existing groups that can not communicate with each

others. This structure has an effect on the worth of coalitions and alter the characteristic function of the underlying transferable utility game (TU-game). [Owen \(1977\)](#) also relies on coalition structures (called a-priori unions) although they directly impact the payoff of the players. In the present article we take yet another direction: this structure restricts the formation of coalitions and therefore their feasibility.

More generally, this paper joins a line of works enriching the standard model of cooperative game theory by considering new structures of cooperation. A large part of this research focuses on structures that restrict cooperation, by modelling either asymmetries in communication or hierarchical constraints. In two papers, Myerson modelled communication by, respectively a graph ([1977](#)) and an hyper-graph ([1980](#)), that both represent a restriction of communication between players. Various solution concepts have been proposed for such games. We can cite for instance the Position-value ([Meessen, 1988](#)), the Average-Tree Value ([Herings et al., 2008](#)), the Mean value ([Hamiache, 2004](#)), the F-value ([Hamiache, 1999](#)), the Hamiache-Navarro value ([Hamiache and Navarro, 2019](#)) and numerous variations of the Myerson value³. Additionally, different forms of hierarchical constraints have been explored by [Gilles et al. \(1992\)](#), [Faigle and Kern \(1992\)](#) and more recently by [Béal et al. \(2021\)](#). The distinction between the framework of [Aumann and Dreze \(1974\)](#), [Owen \(1977\)](#) and ours can similarly be made for, respectively, the framework of [Gilles et al. \(1992\)](#), [Béal et al. \(2021\)](#) and [Faigle and Kern \(1992\)](#). [Béal et al. \(2021\)](#) offer a deeper discussion on how the same mathematical model (in this case, of hierarchical constraint) can affect the overall game in different ways. In this sense, our work is in line with this trend of research.

However, to the best of our knowledge few models have been proposed to enrich the individual characteristics of the players. Another approach to roles, or more precisely differently skilled players, has been proposed by [Bachrach et](#)

³For instances two efficient extensions of the Myerson value were proposed by [Hamiache \(2012\)](#) and [Béal et al. \(2015\)](#). Examples of variations on the theme can be found by [Gómez et al. \(2003\)](#) and [González-Arangüena et al. \(2003\)](#).

al. (2013) with coalitional skill games (CSG). However, in a CSG framework, certain players can perform certain tasks and the worth of a coalition depends on the nature or number of tasks that can be performed by the coalition. The model is thus similar to the standard model of cooperative game theory but restricts itself to a certain type of characteristic functions defined by the players’ “skills”.

Thus, our model tackles issues not addressed by existing research and contributes to the economic theory literature on several levels. For the standard model of TU-games, we provide a new characterization of the egalitarian value. In addition, we use a formal framework of coalition structure to develop a novel class of unexplored problems as of yet. Within this class, we investigate the effect of sharing rules on the spontaneous emergence of coalitions, making it somewhat endogenous to the model. We give a solution for these problems and show that, for this class of problem, egalitarianism can provide incentives whereas approaches based on the evaluation of marginal contributions can be detrimental for cooperation.

In section 2 of this paper we introduce the canonical model of cooperative games with transferable utility and present the axiom of *intersecting population monotonicity* with which we build a new characterization of the egalitarian value. In Section 3 we present our new model of games with roles. In section 4 we propose axioms for a desirable solution to this new type of problems as well as present our main results, which make use of the axiom and result presented in section 2. We study the robustness of our results in section 5. We offer applications of the new model and the results in section 6 before adding some concluding remarks in section 7.

2. Egalitarian value and Intersecting Population Monotonicity

Let U be a non-empty and infinite set of players. A coalition is a non-empty and finite subset of U . A coalitional game with transferable utility (also called a cooperative TU-game) is a pair (N, v) where N is a coalition and v is a function

satisfying $v : 2^N \rightarrow \mathbb{R}$ and $v(\emptyset) = 0$. We note Γ the set of these games. Given a coalition $S \subseteq N$ we write $s = |S|$ its cardinality. For any $S \subsetneq N$ we will write $(S, v|_S)$ the sub-game restricted to coalition S with $v|_S(T) = v(T)$, $\forall T \subseteq S$. A player $i \in N$ is a null player in the game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. A player $i \in N$ is a nullifying player in the game (N, v) if $v(S) = 0$, for all $S \subseteq N$ such that $i \in S$. Two players $i, j \in N$ are equals in the game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A TU-game v is said to be a convex game if it satisfies $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S \subseteq T \subseteq N \setminus \{i\}$. Convex game have been shown by [Shapley \(1971\)](#) to have a non-empty core⁴.

A sharing rule on Γ is a function ψ which associates with each game $(N, v) \in \Gamma$ a vector $\psi(N, v) \in \mathbb{R}^N$. The Shapley value, first presented by [Shapley \(1953\)](#), rewards players based on their marginal contributions, weighted by all the possible orders of entry of players in coalitions. It is noted Sh and defined as

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{i\})], \forall i \in N.$$

The Egalitarian value divides equally the worth of coalition N between all players. It is noted EG and defined as

$$EG_i(N, v) = \frac{v(N)}{n}, \forall i \in N.$$

We introduce the following classical axioms for sharing rules over TU-games.

Efficiency. For each game $(N, v) \in \Gamma$ it holds that $\sum_{i \in N} \psi_i(N, v) = v(N)$.

Linearity. For each couple games (N, v) , $(N, w) \in \Gamma$ and each real number $\alpha \in \mathbb{R}$, it holds that $\psi(N, v + \alpha w) = \psi(N, v) + \alpha \psi(N, w)$.

Null player. For each game $(N, v) \in \Gamma$ and each null player $i \in N$, it holds that $\psi_i(N, v) = 0$.

Nullifying player. For each game $(N, v) \in \Gamma$ and each nullifying player $i \in N$,

⁴Quoting this perfect definition from [Shapley \(1971\)](#), “the core of [a TU-game] is the set of feasible outcomes that cannot be improved upon by any coalition of players”.

it holds that $\psi_i(N, v) = 0$.

Equal treatment of equals. For each game $(N, v) \in \Gamma$ and each couple of players $i, j \in N$ who are equals in (N, v) , it holds that $\psi_i(N, v) = \psi_j(N, v)$.

As shown by [Shubik \(1962\)](#), the Shapley value is the unique sharing rule satisfying efficiency, linearity, null player and equal treatment of equals. The egalitarian value has been shown by [van den Brink \(2007\)](#) to be the unique sharing rule satisfying efficiency, linearity, nullifying player and the equal treatment of equals.

In 1983, [Thomson](#) introduced the axiom of population monotonicity in the context of fair division. This axiom stated that when facing a given problem of fair division, no player should be better off by having to share with more players than less. [Chun \(1986\)](#) and later [Chun and Park \(2012\)](#) expanded on this axiom on, respectively, fair division problem and allocation problems in cooperative games. This weakened version stated that when a population was joined by newcomers, all members of the original population are all either better off or worse off.

We introduce a new axiom of **intersecting population monotonicity** by taking another direction from [Thomson \(1983\)](#). Let us consider a given population P offered with the choice of cooperating with two different groups of the same size. Our axiom states that no member of population P can be worse off if the population chooses the group with whom it has the best synergy. In the context of cooperative games, given two coalition of players whose intersection is non-empty, we ask that no member of the intersection is rewarded less when considering the coalition with the better worth. Formally,

Intersecting population monotonicity. For each game (N, v) with $n \geq 3$ and each couple of coalitions $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$ if $v(S) \geq v(T)$ it holds that $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$, $\forall i \in S \cap T$.

We give an alternative interpretation for this axiom. If a subgroup P of S

decides to replace the other members of S by an identical number of outsiders, then the members of P should not be worse off if the newly formed coalition T is at least as productive as the original coalition S . Note that this axiom puts no requirement on any payoffs in the game (N, v) . Moreover, for the two sub-games considered, it says nothing about the payoffs of players outside the intersection of coalitions S and T . As a result, this axiom is not as strong as it can first appear.

The axiom of *intersecting population monotonicity* is also quite different from the axiom of *grand coalition monotonicity* used by [Casajus and Huettner \(2014\)](#) to characterize the egalitarian value. Grand coalition monotonicity compares the same population involved in two different games, whereas we compare two different populations. Nonetheless some proximity can be seen in the sense that our axiom also provides a monotonic approach to the egalitarian principle, as we will see with our first results.

Lemma 1 *For every game $(N, v) \in \Gamma$ such that all players $i \in N$ are equal in (N, v) , the sharing rule $\psi(N, v)$ satisfies **efficiency and intersecting population monotonicity** if and only if $\psi_i(N, v) = \psi_j(N, v)$, $\forall i, j \in N$.*

Proof. We have a game (N, v) such that all players are equal in v . Let $p \in U \setminus N$, and let us consider a bigger game (N^+, w) such that $N^+ = N \cup \{p\}$ and $w|_N(S) = v(S)$ for all $S \subseteq N$ and such that all players of N^+ are equal in w . This means that $w(S) = c_s$, $\forall S \subseteq N^+$, with $(c_s)_{s=1, \dots, n+1}$ a vector of real numbers. Since U is infinite, such a construction is always possible⁵.

We consider coalition N and we will prove that $\psi_i(N, w|_N) = \psi_j(N, w|_N)$ for all $i, j \in N$. For any two $i, j \in N$, let us consider two additional coalitions $S, T \subsetneq N^+$ of size n in which, respectively, players j and i are replaced with player p . We have

- $S = N \setminus \{j\} \cup \{p\}$,
- $T = N \setminus \{i\} \cup \{p\}$,

⁵If U is finite, this construction is always possible for any N such that $n \leq u - 1$.

- $s = t = n$,
- $S \cap T = N \setminus \{i, j\} \cup \{p\}$,
- $S \cap N = N \setminus \{j\}$,
- $T \cap N = N \setminus \{i\}$.

By definition of w , we have $w(S) = w(T) = v(N)$. Such a construction of S and T is always feasible if $n \geq 3$. Since the worths of coalitions S , T and N are equal, we use *intersecting population monotonicity* in both directions and obtain the following useful conditions on ψ :

$$\begin{aligned}\psi_k(S, w|_S) &= \psi_k(T, w|_T), \quad \forall k \in S \cap T, \\ \psi_i(S, w|_S) &= \psi_i(N, w|_N), \\ \psi_j(T, w|_T) &= \psi_j(N, w|_N).\end{aligned}$$

By *efficiency* we have the following condition on ψ :

$$\sum_{k \in S} \psi_k(S, w|_S) = w(S) = w(T) = \sum_{k \in T} \psi_k(T, w|_T).$$

Therefore we obtain that

$$\begin{aligned}\sum_{k \in S \cap T} \psi_k(T, w|_T) &= \sum_{k \in S \cap T} \psi_k(S, w|_S) \\ \Leftrightarrow \sum_{k \in S \cap T} \psi_k(T, w|_T) + \psi_i(S, w|_S) &= \sum_{k \in S \cap T} \psi_k(S, w|_S) + \psi_i(S, w|_S) \\ \Leftrightarrow \sum_{k \in S \cap T} \psi_k(T, w|_T) + \psi_i(S, w|_S) &= w(S) \\ \Leftrightarrow \sum_{k \in S \cap T} \psi_k(T, w|_T) + \psi_i(S, w|_S) &= \sum_{k \in S \cap T} \psi_k(T, w|_T) + \psi_j(T, w|_T) \\ &\Leftrightarrow \psi_i(S, w|_S) = \psi_j(T, w|_T) \\ &\Leftrightarrow \psi_i(N, w|_N) = \psi_j(N, w|_N).\end{aligned}$$

This proves that for any two players $i, j \in N$ we have $\psi_i(N, w|_N) = \psi_j(N, w|_N)$ which, by definition of w , is equivalent to $\psi_i(N, v) = \psi_j(N, v)$.

□

Theorem 1⁶

For every game $(N, v) \in \Gamma$, the sharing rule $\psi(N, v)$ satisfies **efficiency** and **intersecting population monotonicity** if and only if $\psi = EG$.

Proof. We know that EG satisfies efficiency. It is trivial that it satisfies intersecting population monotonicity. Let us prove that, for games $(N, v) \in \Gamma$ if a sharing rule ψ satisfies efficiency and intersecting population monotonicity then it is the egalitarian value.

Let us consider a game (N, v) and a player $i \in N$. We can build a game (N_i, w_i) with N_i such that

$$\begin{aligned} N &\subsetneq N_i, \\ \exists T &\subsetneq N_i \text{ with } N \cap T = \{i\} \text{ and } n = t. \end{aligned}$$

We also put the following conditions on w_i :

$$\begin{aligned} w_i(P) &= v(P), \quad \forall P \subseteq N \\ w_i(T) &= v(N), \\ w_i(P) &= v(\{i\}), \quad \forall P \subsetneq T, \quad P \neq \{i\}. \end{aligned}$$

Such a construction is always possible since U is infinite.⁷ By definition we have $(w_i)_{|N}(S) = v(S)$ for all $S \subseteq N$ hence $\psi_i(N, (w_i)_{|N}) = \psi_i(N, v)$. Using efficiency and lemma 1 we have

$$\psi_i(T, (w_i)_{|T}) = \frac{(w_i)_{|T}(T)}{t} = \frac{v(N)}{n}$$

since all players $j \in T$ are equals in $(w_i)_{|T}$. As we have $N, T \subseteq N_i$, $n = t$ and

⁶If we consider a finite universe of player U , this result holds for any N such that $n \leq \frac{u+1}{2}$.

⁷If U is finite, such a construction is possible for any N such that $n \leq \frac{u+1}{2}$.

$w_i(T) = w_i(N)$ using intersecting population monotonicity we obtain that

$$\begin{aligned} \psi_i(T, (w_i)_{|_T}) &\geq \psi_i(N, (w_i)_{|_N}) \\ \text{and } \psi_i(N, (w_i)_{|_N}) &\geq \psi_i(T, (w_i)_{|_T}) \end{aligned}$$

which means

$$\psi_i(N, (w_i)_{|_N}) = \psi_i(T, (w_i)_{|_T}) = \frac{v(N)}{n} = \psi_i(N, v).$$

Since U is an infinite set it is possible to build a game (N_i, w_i) that satisfies our above conditions for every $i \in N$. Hence we have that $\psi_i(N, v) = \frac{v(N)}{n}$, $\forall i \in N$, which concludes our proof. □

We now prove that our three axioms are logically independent.

Efficiency

Consider the sharing rule $f_1(N, v) = \alpha EG(N, v)$ with $\alpha \neq 1$. It obviously satisfies all axioms but efficiency.

Intersecting population monotonicity

Consider the Shapley value. It satisfies efficiency and equal treatment of equals. Application 2 in section 6 shows it does not satisfy intersecting population monotonicity.

3. Games with roles

We develop here a new model of cooperative games in which players have a specific role they can play in the coalitions.

Consider an entity (a firm for instance) is setting up a project that asks for specific roles to be performed. It has several players (candidates, employees, other firms) that can fill these roles. Each player can only assume a specific role but several players are able to play the same role. Once the team is put together, the player that are parts of the team will work on the project which

implies working all together as well as in subgroups and alone.

Formally, we consider k roles with $1 < k < n$. Each player $i \in N$ can perform only one specific role and we note $K_\alpha \subsetneq N$ the set of players able to perform the role $\alpha \in \{1, \dots, k\}$. We write $K = \{K_1, K_2, \dots, K_k\}$ the set of those sets. We write K^i the set of players able to perform the same role as player i . Noting $\mathcal{P}(N)$ the partitions of the player set N , the set $K \in \mathcal{P}(N)$ can be understood as a coalition structure as introduced by [Aumann and Dreze \(1974\)](#). We have

$$\bigcup_{1 \leq \alpha \leq k} K_\alpha = N \text{ and } K_{\alpha_1} \cap K_{\alpha_2} = \emptyset \text{ for any } \alpha_1, \alpha_2 \in \{1, \dots, k\} \text{ and } \alpha_1 \neq \alpha_2.$$

We name *full coalition* a coalition where each role is filled. We note $F = \prod_{1 \leq \alpha \leq k} K_\alpha$ the set of full coalitions, defined by the cartesian products of the sets that partition players into roles. Therefore, the size of each full coalition is k . We also note $F^i = \{F_l \in F \mid i \in F_l\}$ the set of full coalitions to which player i belongs. We assume that sub-coalitions of a full coalitions are still feasible but that coalitions with excess players are not⁸. Hence the set of feasible coalitions $\mathcal{F}_{N,K}$ is the union of the subsets of full coalitions.

We consider that a value is generated from the cooperation of players. This production is modelled by a characteristic function v defined over 2^N . Hence, we define a game with roles as a triplet (N, v, K) with $N \subseteq U$, $(N, v) \in \Gamma$ and $K \in \mathcal{P}(N)$. We note Γ_{role} the set of those games.

This model naturally raises three issues: the formation of one of the full coalitions, the allocation of its worth, and the influence of this allocation on the stability of the formation process of the selected full coalition. To address these issues, a solution on Γ_{role} is a pair $\phi = (\mu, \psi)$. The function μ is a mechanism that selects for each $(N, v, K) \in \Gamma_{role}$ one of its full coalitions $\mu(N, v, K)$ and ψ is a sharing rule on Γ .

⁸As actors in a play have scenes with each other as well as monologues and a full-house grand finale, our players working on a project have tasks to perform in subgroups or alone.

4. Optimality and outside options

A solution $\phi = (\mu, \psi)$ on Γ_{role} consists of a coalition selector μ and a sharing rule ψ and is therefore two-dimensional. As such we will put requirements on both dimensions separately as well as a requirement on their interaction.

One desirable and reasonable requirement for a mechanism μ would be that it selects an optimal full coalition. A full coalition $F_l \in F$ is optimal if and only if $v(F_l) = \max_{F_p \in F} v(F_p)$. We note $F_{opt} = \{F_l \in F \mid v(F_l) = \max_{F_p \in F} v(F_p)\}$ the set of optimal full coalitions. Hence, we ask that the mechanism μ selects a coalition that is optimal⁹. This gives us a first axiom:

Optimality. For every games with roles $(N, v, K) \in \Gamma_{role}$ it holds that $\mu(N, v, K) \in F_{opt}$.

Next we invoke the standard axiom of **efficiency** (as defined in section 2) for the sharing rule ψ on Γ .

Finally, we introduce an axiom that concerns the interaction between μ and ψ . Given a solution $\phi = (\mu, \psi)$, can we make sure that $\mu(N, v, K)$ is stable? That is to say, given ψ , can we make sure that no players in $\mu(N, v, K)$ would be better off associating with other players? We consider the payoff a player can obtain with another coalition as a player's outside options. Hence the following question: does the solution $\phi = (\mu, \psi)$ is such that no player that belongs to $\mu(N, v, K)$ has better outside options?¹⁰ Formally:

No outside options. For every game with roles $(N, v, K) \in \Gamma_{role}$ it holds that $\phi = (\mu, \psi)$ is such that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F_p, v_{|F_p}), \forall F_p \in F^i, \forall i \in \mu(N, v, K)$.

In the following two results we explore the relationship between *optimality*,

⁹Note that there can be multiple optimal full coalitions. We only ask that μ selects one of these. Hence it is possible that μ is not anonymous.

¹⁰The concept of outside options considered here is in line with the one used by Casajus (2009) in the context of communication graph games.

no outside options and intersecting population monotonicity.

Theorem 2

Let us have a game with role $(N, v, K) \in \Gamma_{role}$ and a solution $\phi(\mu, \psi)$. If μ satisfies **optimality** then ϕ satisfying **no outside options** is equivalent to ψ satisfying **intersecting population monotonicity**.

Proof. We first prove that if μ satisfies **optimality** and ψ satisfies **intersecting population monotonicity** then ϕ satisfies **no outside options**.

Let us consider a solution $\phi = (\mu, \psi)$. If μ satisfies *optimality* then $\mu(N, v, K) \in F_{opt}$ which implies that $v(\mu(N, v, K)) \geq v(F_p)$ for all $F_p \in F$. By definition we have that $|\mu(N, v, K)| = |F_p|$, for all $F_p \in F$. Using *intersecting population monotonicity* we know then that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)|}) \geq \psi_i(F_p, v_{|F_p|})$ for all F_p such that $\mu(N, v, K) \cap F_p \neq \emptyset, \forall i \in \mu(N, v, K) \cap F_p$. By definition, if $i \in \mu(N, v, K) \cap F_p$ then $F_p \in F^i$. Hence $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)|}) \geq \psi_i(F_p, v_{|F_p|})$ for all $F_p \in F^i$, for all $i \in \mu(N, v, K)$. This coincides with the requirement for *no outside options*.

We now prove that if μ satisfies **optimality** and ϕ satisfies **no outside options** then ψ satisfies **intersecting population monotonicity**.

Let us consider a solution $\phi = (\mu, \psi)$. If μ satisfies *optimality* then $\mu(N, v, K) \in F_{opt}$ which implies that $v(\mu(N, v, K)) \geq v(F_p)$ for all $F_p \in F$. Using *no outside options* we know that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)|}) \geq \psi_i(F_p, v_{|F_p|}), \forall F_p \in F^i, \forall i \in \mu(N, v, K)$. All $F_p \in F^i$ for a given $i \in \mu(N, v, K)$ are all $F_p \in F$ such that $i \in \mu(N, v, K) \cap F_p$. Hence the requirement for *no outside options* becomes $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)|}) \geq \psi_i(F_p, v_{|F_p|}), \forall F_p \in F, F_p \cap \mu(N, v, K) \neq \emptyset, \forall i \in F_p \cap \mu(N, v, K)$. This requirement must be true $\forall K \in \mathcal{P}(N)$. Whatever $K \in \mathcal{P}(N)$ is considered we always have $F_p \subseteq N, \mu(N, v, K) \subseteq N, |F_p| = |\mu(N, v, K)|$ for all $F_p \in F, F_p \cap \mu(N, v, K) \neq \emptyset$. This being true for all K means that, given a certain TU-game $(N, v) \in \Gamma$, for every two coalitions

$S, T \subseteq N$ with $S \cap T \neq \emptyset$, $s = t$ we have the following requirement on ψ :

$$\psi_i(S, v|_S) \geq \psi_i(T, v|_T), \quad \forall i \in S \cap T, \text{ whenever } v(S) \geq v(T).$$

This coincides with *intersection population monotonicity* for ψ . □

Theorem 3

For every game with role $(N, v, K) \in \Gamma_{role}$ such that $k \leq \frac{n+1}{2}$. Consider the solution $\phi(\mu, \psi)$ with μ satisfying **optimality** and ψ satisfying **efficiency**. The solution ϕ satisfies **no outside options** if and only if $\psi = EG$.

Proof. We first prove the existence part. We consider a solution $\phi(\mu, EG)$ with μ satisfying *optimality*. We know that *EG* satisfies *efficiency* and *intersecting population monotonicity*. From theorem 2 we conclude that $\phi(\mu, EG)$ satisfies *no outside options*.

Now the unicity part. We consider a solution $\phi(\mu, \psi)$ with μ satisfying *optimality* and ψ satisfying *efficiency*. Since ϕ satisfies *no outside options*, from theorem 2 we know that ψ satisfies *intersecting population monotonicity*. Given the game (N, v, K) , every full coalition is of size k . Since $k \leq \frac{n+1}{2}$ we can use theorem 1 to conclude that $\psi = EG$. □

5. Robustness

In this section we study the robustness of our results by looking at different variations of the *no outside options* axiom.

The definition of the axiom of *no outside option* can appear quite restrictive. We previously defined the *no outside option* axiom on the following idea of coalitional stability: a coalition can only form if each player consents to its

formation. This conception is in line with the model 1 of stability presented in [Hart and Kurz \(1983\)](#). Additionally, this conception can also be interpreted as a requirement for a deviation. Therefore, in this sense, coalitional stability can be thought as the inability for players to form another coalition than the one they are currently in. This second approach would give us the following axiom:

No outside options (weak). *For every game with roles $(N, v, K) \in \Gamma_{role}$, it holds that the solution $\phi(N, v, K) = (\mu, \psi)$ is such that $\exists i \in \mu(N, v, K) \cap F_p$ such that*

$$\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_i(F_p, v_{|F_p}), \quad \forall F_p \in F.$$

An alternative coalition F_p can form only if no member of $\mu(N, v, K) \cap F_p$ object to its formation.

Now, let us consider the following axiom defined for sharing rule over TU-games:

Intersecting population monotonicity (weak). *For each game $(N, v) \in \Gamma$ with $n \geq 3$ and each couple of coalitions $S, T \subseteq N$ with $s = t$, $S \cap T \neq \emptyset$ if $v(S) \geq v(T)$ it holds that $\exists i \in S \cap T$ such that $\psi_i(S, v_{|S}) \geq \psi_i(T, v_{|T})$.*

This axiom is sufficient to replace the *intersecting population monotonicity* in the proof of theorem 1. Theorem 2 establishes a link between the axiom of *intersection population monotonicity* for sharing rule on TU-games and *no outside options* for solutions on games with role, when μ satisfies *optimality*. Similarly we can write the following theorem:

Theorem 4

*Let us have a game with role $(N, v, K) \in \Gamma_{role}$ and a solution $\phi(\mu, \psi)$. If μ satisfies **optimality** then ϕ satisfying **no outside options (weak)** is equivalent to ψ satisfying **intersecting population monotonicity (weak)**.*

This result draws the same link between *no outside options (weak)* and *intersecting population monotonicity (weak)*. The proofs for this theorem is omitted as it is quite similar to its counterpart. This implies that theorem 3 still holds when we replace *no outside options* by this weaker version.

Going in another direction, we can build a more restrictive version of the *no outside option* axiom. We require that the payoff obtained by each player with the chosen full coalition $\mu(N, v, K)$ should be at least as large as the payoff that any other player able to perform the same role can obtain in any other full coalition. Application 1 in section 6 illustrate a situation in which this requirement is relevant.

Therefore, we introduce the following axiom:

No outside options (strong). *For every game with roles $(N, v, K) \in \Gamma_{role}$ it holds that $\phi = (\mu, \psi)$ is such that $\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) \geq \psi_j(F_p, v_{|F_p})$, $\forall F_p \in F^j, \forall j \in K^i, \forall i \in \mu(N, v, K)$.*

In this stronger version we compare the payoff of a player i of the selected full coalition $\mu(N, v, K)$ with every payoff of every other players able to perform the same role as i in every full coalitions. It is clear that this axiom is a stronger version of *no outside options* and that *no outside options (strong)* implies *no outside options*. We now prove the following result.

Theorem 5

*For every game with role $(N, v, K) \in \Gamma_{role}$ such that $k \leq \frac{n+1}{2}$. Consider the solution $\phi(\mu, \psi)$ with μ satisfying **optimality** and ψ satisfying **efficiency**. The solution ϕ satisfies **no outside options (strong)** if and only if $\psi = EG$.*

Proof. We first prove the existence part. Consider a solution $\phi(\mu, EG)$ with μ satisfying *optimality*. We know that EG satisfies *efficiency*. We have

$$\psi_i(\mu(N, v, K), v_{|\mu(N, v, K)}) = \frac{v_{|\mu(N, v, K)}(\mu(N, v, K))}{k}, \quad \forall i \in \mu(N, v, K),$$

as well as

$$\psi_j(F_p, v_{|F_p}) = \frac{v_{|F_p}(F_p)}{k}, \quad \forall F_p \in F^j, \quad \forall j \in N.$$

Since μ satisfies *optimality* we have $v_{|\mu(N, v, K)}(\mu(N, v, K)) \geq v_{|F_p}(F_p) \quad \forall F_p \in F$. Hence, $\phi(\mu, ED)$ satisfies *no outside options (strong)*.

Now the unicity part. We consider a solution $\phi(\mu, \psi)$ with μ satisfying *optimality* and ψ satisfying *efficiency*. Since ϕ satisfies *no outside options (strong)*, it satisfies *no outside options*. From theorem 2 we know that ψ satisfies *intersecting population monotonicity*. Given the game (N, v, K) , every full coalition is of size k . Since $k \leq \frac{n+1}{2}$ we can use theorem 1 to conclude that $\psi = EG$. □

6. Applications

In this section we illustrate our results by applying our model to rating systems in online gaming (application 1), to organizational settings (application 2) and to resource management (application 3). We intend to show by these applications that our model can tackle widely different issues. Application 1 illustrate both the model and our results in an already used mechanism. Application 2 provide another context where the model can be used, and show how the Shapley value would behave in this context. Application 3 only describe how the model could be used to describe a water sharing problem.

Application 1 (*Team Matchmaking Rating (MMR)*)

In modern team-based competitive video games, players from over the world team up with strangers to compete against other strangers.

Outside of tournaments and organized competitions where professional teams oppose each other, players that want to play a game join a queue of waiting players. These players are then matched by a match-making algorithm. These algorithms match players based on various informations, one of which is a skill rating. These ratings are updated after each games. Usually, a win leads to an increase in rating and a loss to a decrease in rating.

The design of skill ratings is the subject of an active literature (see for instance [Chen et al. \(2017\)](#), [Minka et al. \(2018\)](#)) especially in the context of games opposing teams of players. According to [Minka et al. \(2018\)](#), incentives for cooperative behaviours are a concern of developers when designing their rating system, mainly because the rating is an evaluation of a player’s level of competency and, as such, often becomes a long term objective for players.

One popular genre of team-based competitive video games are the multi-player online battle arena games (or MOBA games, for short). The MOBA game *Dota 2*, developed by Valve Corporation, is currently the biggest video game in terms of competitive earnings¹¹. In this game, two teams of five players must defend their own base while destroying the base of the other team. The game is played on a symmetrical map. Every player independently controls a character. Every character has specific abilities and a different play style. There are currently 124 different characters to pick from. One of the key strategic element is therefore the synergy between the different characters and their style of play.

In the context of our model, each player in the video-game can be represented by a role K_α . Each player can pick a different character, represented in our model by a player $j \in K_\alpha$. We are interested in a rating system that shares the value of the outcome of a match between team members in such a way that players are encouraged to form the best team of characters.

Dota 2 rating system used to be a modified ELO system until april 2023

¹¹*Dota 2* cumulated a total prize pool of 32,954,188.52\$ in 2022, across 125 competitive events according to the website [esportsearnings.com](https://www.esportsearnings.com). Source: <https://www.esportsearnings.com/history/2022/games>, consulted on the 11/7/2023.

where it was replaced by a modified Glicko system (a bayesian variation of the ELO system). These two systems were initially developed for chess rankings. Compared to other rating system used in competitive video games such as the TrueSkill algorithm (Minka et al., 2018), they do not measure the individual performance of a player. When a team wins (or loses), every member of a team is given the same bonus (or malus) to its rating.

The way these two systems used for *Dota 2* allocate the value of a match's outcome between team members is egalitarian since every member gets the same change in rating. This system is not always well received by players¹². Players can have a tendency to overestimate their skills, and blame a loss on their teammates. However, theorem 5 shows that this egalitarian approach is the only one where players always have an incentive to form the most efficient team (or what they think would be the most efficient team). In particular, a system relying on a measure of individual performance would lead players to try and maximize their performance according to this measure, possibly to the detriment of the one obvious common goal: a win for the team.

Hence, while the system that Valve Corporation adopted for *Dota 2* is already easier to design and compute than a complex system that would try to capture a individual performance within a team effort, our results show that it also has the benefit of a strong incentive to behave in line with the common goal.

Application 2 (*Team project*)

Consider the following situation: a business trusts one of its managers to take on a new project that needs two other specialists. She is given freedom to choose other employees to fill these roles. The other employees are free to reject the proposition. Once the project is done, the team will get a bonus proportional to the estimated value of the work produced. This raises the problem

¹²According to Valve Corporation, "Players' appraisals of matchmaking quality are highly correlated with their recent win rate". Source: <https://web.archive.org/web/20131224214430/http://blog.dota2.com/2013/12/matchmaking/>, consulted on 11/7/2023.

of finding a sharing rule for this bonus that gives the manager the incentives to put together the team that produce the most valuable work. Formally we have $N = \{1, 2, 3, 4, 5\}$ and the role structure $K = \{\{1\}, \{2, 3\}, \{4, 5\}\}$. The full coalitions are $F = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}\}$. The set of feasible coalition is given by

$$\mathcal{F}_{N,K} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}\}.$$

Imagine that 2 has poor synergy with both 4 and 5 while 3 has a really good synergy with 5 (only amplified under 1 management) but can't stand working with 4. However, as a manager, 1 is able to be at its best when bridging the divide between 4 and 3. The following worths for the feasible coalitions express this situation¹³. The worths express the value of intermediary works as well as the end product. We have $v(\{i\}) = 0$ for all $i \in N$ and

S	12	13	14	15	24	25	34	35	124	125	134	135
v(S)	0.5	0	1	1	0.5	0.75	0	2	2	2	3	4

Let us first consider a sharing rule based on marginal contribution: the Shapley value. It gives the following results for each full coalitions

Full coalition $F_p \in F$	124	125	134	135
$Sh_1(F_p, v _{F_p})$	0.75	~ 0.666	~ 1.166	~ 0.833
$Sh_2(F_p, v _{F_p})$	0.5	~ 0.541	-	-
$Sh_3(F_p, v _{F_p})$	-	-	~ 0.666	~ 1.333
$Sh_4(F_p, v _{F_p})$	0.75	-	~ 0.666	-
$Sh_5(F_p, v _{F_p})$	-	~ 0.791	-	~ 1.833

We can see that the Shapley value does not satisfy *intersecting population monotonicity*. The solution $\phi(\mu, Sh)$ with μ satisfying *optimality* does not satisfy *no outside options*. The full coalition giving the highest worth is $\{1, 3, 5\}$ but 1

¹³Note that in this example, for any $F_p \in F$ the sub-game $v|_{F_p}$ is convex.

would prefer that 3 and 4 accept to work on her team. Similarly, 4 would rather work with 2 than with 3. In each full coalitions, at least one player has an outside options when the solution ϕ uses the Shapley value.

Across the organization, using a solution $\phi(\mu, EG)$ with μ satisfying optimality ensures that the most effective team will be formed willingly.

Application 3 (*Water sharing*)

Finally we propose an application of the model to a public economy problem. Let us consider a water source with N potential consumers. The consumption time is broken into k time periods $1, \dots, k$. Each consumer only need water in one period. We note K_α the set of agents wanting to consume water at period α for $\alpha \in \{1, 2, \dots, k\}$. $K = \{K_1, K_2, \dots, K_k\}$ is the induced partition on N . Every potential consumer $i \in N$ has a need $b_i > 0$ and each period k has a maximum capacity of C_{K_k} . We want to choose the set of consumers such that there is no overconsumption of the resource while the least amount of water goes unused. Hence the worth of a coalition $S \subseteq N$ is given by

$$v(S) = \sum_{\alpha \in \{1, \dots, k\}} \max_{i \in S \cap K_\alpha} \{ \min(b_i; C_{K_\alpha}) \},$$

and is the feasible total consumption of coalition S that leaves the least amount of water unused.

7. Concluding remarks

In this paper we developed a model of cooperative environments in which multiple roles have to be performed. This model is built on the canonical model of cooperative TU-games. We showed that, if the number of players is sufficiently larger than the number of roles, an egalitarian approach to the allocation of the outcome of cooperation is the only efficient approach that always provide incentive for the best coalition to form. We showed that this result is still true when a weaker or stronger definition of stability is considered.

In our approach, the full domain of cooperative games is considered. Application 2 shows that the solution $\phi = (\mu, Sh)$ does not satisfy no outside options even for convex games. On specific classes of games, it would be interesting to determine whether other sharing rule than the egalitarian value are compatible with the axiom of no outside options

In addition, it is well known that the payoff vector given by the egalitarian value often lies outside the core, even for convex games. Consequently the selected full coalition can offer no outside options in the sense that all players involved prefer working together than within another full coalition but not be stable in the sense that this cooperation might fail. As such our result could also be framed as an impossibility result: there are no solution $\phi = (\mu, \psi)$ with μ satisfying *optimality* and ϕ satisfying *no outside options* such that $\psi(\mu, v_{|\mu})$ lies in the core of $(\mu, v_{|\mu})$. However, the fact that ψ might not be in the core can possibly be irrelevant if we consider situations where the cooperation is based on free association but, once established, is subject to binding agreements. The applications we have presented show that these situations can exist in different contexts, on which our result can provide insights.

In conclusion, on the basis of the main result of this paper, we argue that egalitarianism is the only approach that provides incentives towards optimality for every cooperative situations that satisfies the following conditions:

- cooperation is based on free association;
- the numbers of players needed is sufficiently smaller than the numbers of candidates.

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