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PII: S0951-8320(22)00532-4
DOI: https://doi.org/10.1016/j.ress.2022.108917
Reference: RESS 108917

To appear in: *Reliability Engineering and System Safety*

Received date: 8 June 2022
Revised date: 14 September 2022
Accepted date: 20 October 2022

Please cite this article as: Weian Yan , Xiaofan Xu , David Bigaud , Wenqin Cao , Optimal design of step-stress accelerated degradation tests based on the Tweedie Exponential Dispersion Process, *Reliability Engineering and System Safety* (2022), doi: https://doi.org/10.1016/j.ress.2022.108917

Highlights

- The Tweedie exponential dispersion process model based on SSADT is generalized.
- The optimum plan is the one that uses only the minimum and maximum stress levels.
- An explicit expression of allocation proportion is given.
- The proposed optimum plan can improve the performance through validation.

Optimal design of step-stress accelerated degradation tests based on the Tweedie Exponential Dispersion Process

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Abstract: In this paper, we are interested in the optimization of step-stress accelerated degradation test plan when the degradation path follows the Tweedie exponential dispersion process which has been shown to be an important family in degradation analysis. We firstly prove that, under the Tweedie exponential dispersion process model with a drift parameter being an exponential function of the (transformed) stress level, a multi-level step-stress accelerated degradation test plan will correspond to a simple step-stress accelerated degradation test plan using only the minimum and maximum stress levels under D-optimality and V-optimality criteria. The optimum step-stress accelerated degradation test plan based on these two optimality criteria is subsequently derived, a numerical example is furthermore presented to compare the efficiency of the proposed optimum simple step-stress accelerated degradation test plans and some step-stress accelerated degradation test plans proposed by a previous study. In addition, a simulation study is conducted for investigating the performances of the proposed step-stress accelerated degradation test plans.

Keywords: Reliability, Step-Stress Accelerated Degradation Test, Tweedie Exponential Dispersion Process, Optimal Design

1. Introduction

Assessing the reliability information of product is an essential task for manufacturers, especially for newly developed products [1]. However, for highly reliable products, it is not an easy task to obtain their life information by using traditional life test because failures are not likely to occur in a short and acceptable duration even by censoring life test [2]. Usually, the degradation of a product is also very slow under normal use or service conditions. In addition, for some expensive products, the economic cost of obtaining enough life data is too high or even unbearable. To obtain the reliability information more rapidly/efficiently, we can conduct an accelerated degradation tests (ADT) campaign that uses higher levels of factors accelerating the quality characteristics degradation (e.g., factors or variables such as temperature or voltage) to obtain the lifetime information at the normal use or service condition using extrapolation [3].

ADT has become an efficient approach to reliability assessment or lifetime prediction for degrading products [4]. Depending on the different stress loadings application, ADT can be classified into constant-stress accelerated degradation test (CSADT), step-stress accelerated degradation test (SSADT) and progressive-stress accelerated degradation test [5]. To fully get the potential of the subsequent analysis of ADT results, we must carefully design the ADT to ensure an accurate evaluation of reliability-related indexes at the service condition and maximize the efficiency-to-cost ratio. Therefore, the optimal design problem of an ADT plan received considerable attention from reliability researchers and engineers [6]. In a CSADT, the products are divided into several groups, and each group of products are exposed to a constant severe stress condition to collect degradation data. Tsai et al. [7] discussed the optimal design problem of the CSADT based on the Gamma process. They determined the test stress levels and the proportion of units allocated to each stress level by minimizing the asymptotic variance of the mean time to failure. Wang et al. [8] developed an optimum CSADT plan for inverse Gaussian (IG) process model under M-optimality criterion from the perspective of degradation mechanism equivalence. Jiang et al. [9] obtained the optimum CSADT plan for Wiener process model based on the criterion that minimize the mean of the upper prediction limit for the degradation characteristic at the use condition. More researches on CSADT optimization can be found in [10-11].

Although CSADT is an efficient test, it needs a large number of units to conduct an experiment [12]. For a newly developed product or an expensive product, we may not have many test units on hand. In this situation, SSADT is more suitable to be adopted since it requires less test units [13]. Besides, it has been shown that using step-stress stress loading can provide equivalent estimation precision to that from other stress loadings [14]. Furthermore, it has been shown that there exist SSADT plans that can generate a Fisher information matrix identical to that derived from a general stress loading function [15]. Given these advantages of SSADT, extensive studies have been conducted to obtain optimum SSADT plans. Duan and Wang [16] discusses the design problem of the SSADT based on the non-stationary Gamma process with random effects. They obtain an optimal experiment plan by minimizing the asymptotic variance of the estimated reliability of the product under the budget and boundary constraints. Zheng and Chen [17] developed an optimal design method for SSADT based on the generalized Wiener process degradation model. Wang et al. [18] Provided an optimal SSADT plan for the IG degradation process under the constraint of the total experimental budget by minimizing the asymptotic variance of the estimated p-quantile of the lifetime distribution of the product. Further applications of SSADT can be found in [19-20].

To carry out the optimal design of SSADT, two critical tasks need to be dealt with, one is the selection of the optimization criterion, and the other is the selection of the degradation model. For the first one, extensive studies have applied different optimization criteria to obtain optimal ADT plans,

and these strategies can be sorted into two categories [17]. The first strategy is to enhance the estimation accuracy of the unknown parameters of degradation models, such as D-optimality. The second strategy focuses on the accuracy of reliability indexes, such as the *p*-quantile of the lifetime distribution, which is called V-optimality and is commonly used in the optimal design of ADTs. Therefore, in this research, the optimal design is conducted based on the D-optimality and V-optimality, respectively.

For the second one, a suitable degradation model is required for analyzing the observed degradation data and to estimate the product's lifetime under use or service condition. There are three commonly used stochastic process models: Wiener process, Gamma process, and IG process [21]. The Wiener process is usually used to describe non-monotonic degradation paths [22], while the Gamma process and IG process models are applied for monotonic degradation paths [23-24]. These three well-known stochastic process models could fit most of degradation data well. However, in some engineering applications, these three well-known processes are not suitable. For example, a discrete-type compound Poisson process may be more appropriate to model a leakage current of thin gate oxides in nanotechnology [25]. Moreover, a degradation model is selected through Akaike information criterion (AIC) from the candidate models. As we know, the drawback of AIC to choose a model is that if all the candidate models fit poorly, AIC will not give any warning of that. Then the corresponding results of the reliability analysis based on the selected unsuitable models could be poor. Hence, considering the diversity of the products, a more general class of degradation model, which has a wide range of applications, is necessary for describing the real degradation data more accurately. To promote the adaptability of the modeling method for degradation data, the Tweedie exponential dispersion process (TED) process was proposed to describe the degradation process of some products, which includes Wiener process, Gamma process, and IG process as special cases. The TED has been proven to provide more suitable degradation models to describe the product's degradation path [26-27]. Zhou and Xu [28] proposed to use the TED process to describe the degradation path of the product's physical or chemical characteristics. They used the maximum likelihood estimation (MLE) and Bootstrap method to obtain the point estimates and interval estimates of the parameters. In the data analysis, they found that the TED model can fit the data much better than the commonly used models. Lee and Tseng [29] proposed a semi-analytical procedure to determine the total sample size, testing stress levels, the measurement frequencies, and the number of measurements (within a degradation path) globally under the TED degradation model. Chen and Xia [30] obtained the optimal CSADT plan based on the numerical computation under the constraint of budget by using TED process. Although the research efforts, including the articles cited above, have been devoted to statistical inference and optimal design of CSADT for the TED process, the SSADT experiment

design problem under TED process has not been discussed to our knowledge. Furthermore, most of the optimal design results of ADT are obtained based primarily on numerical studies, not exact results.

The aim of this research is to develop the method to find the optimal SSADT plan for products with TED degradation process. Some theoretical results are given for these problems. Compared with the existing works, the major contribution of this study lies in the following two aspects: 1) It is proved that, under the TED model with a drift parameter being an exponential function of the (transformed) stress level, a multi-level SSADT plan will degenerate to a simple SSADT plan using only the minimum and maximum stress levels under D-optimality and V-optimality criteria; 2) The optimum number of inspection allocation proportions for each stress level are derived based on these two optimal criteria.

The remainder of this study is organized as follows. In Section 2, the TED model, and the SSADT settings are introduced. In Section 3, we prove that the optimal SSADT plan is indeed a simple SSADT plan using only the minimum and maximum stress levels, and then the optimum SSADT plan are derived. In Section 4, a numerical example is provided to compare the efficiency of a SSADT plan proposed by a previous study and the optimum plans proposed in this paper. In Section 5, a simulation study is conducted for investigating the performance of optimum SSADT plans. Section 6 concludes the paper.

2. Model Description and Statistical inference

In this section, the theoretical frame of SSADT based on the TED degradation process is proposed. At first, we introduce the concepts and some properties of the TED process, and then present some assumptions to describe the SSADT experiment.

2.1 TED Degradation Model

A stochastic process describing the evolution of a performance indicator or any quality characteristics over time is defined as an exponential dispersion (ED) process $\{Y(t), t \geq 0\}$, if satisfying the following three properties [28-31]:

(1) $Y(0) = 0$ with probability one;

(2) $\{Y(t), t \geq 0\}$ has statistically independent increments;

(3) The increment follows ED distribution, i.e., $Y(t + \Delta t) - Y(t) \sim ED(\eta \Delta t, \lambda)$, for $\forall \Delta t > 0$, where the probability density function (PDF) of ED distribution $ED(\mu t, \lambda)$ is

$$f(y \mid t, \eta, \lambda) = c(y \mid t, \lambda) \cdot exp\{\lambda[y\omega(\eta) - t\kappa(\omega(\eta))]\}, \quad (1)$$

where η is the mean drift rate and λ is the dispersion parameter; $c(\cdot)$ is a canonical function, guaranteeing that the cumulative distribution function (CDF) of Equation (1) is normalized and equal to one; $\kappa(\cdot)$ is called the cumulant function, which is a twice differentiable function, and satisfying $\kappa'(\omega(\eta)) = \eta$, in which $\kappa'(\cdot)$ is the first derivative of $\kappa(\cdot)$.

The expectation and the variance of $Y(t)$ are $E\big(Y(t)\big)=\eta t$, $Var\big(Y(t)\big)=\kappa''\big(\omega(\eta)\big)t/\lambda \triangleq V(\eta)t/\lambda$, respectively, in which $\kappa''(\cdot)$ is the second-order derivatives of $\kappa(\cdot)$, and $V(\eta)$ is called the variance function. An ED model can be characterized by its variance function within the class of all ED models. Furthermore, the TED process is an important class of ED process with power variance function:

$$V(\eta)=\eta^{\rho}, \rho \in (-\infty,0]\cup[1,\infty] \tag{2}$$

where ρ is the power classification parameter, which is determined by the degradation mechanism of the product and can be estimated based on degradation data. Different parameters ρ correspond to different stochastic process models, such as the Wiener process ($\rho=0$), the Gamma process ($\rho=2$), and the IG process ($\rho=3$). More details can be found in Table 1. That is, the TEDP model can describe more complex and diverse degradation processes of many products and have a wider range of applicability compared with others stochastic models. For convenience, the TED process is denoted by $Y(t)\sim TED(\mu t,\lambda)$.

Table 1 The relationship between the TED process and some well-known processes

Model	ρ	$V(\eta)$	Mean	Variance
TED	$\rho \in (-\infty,0]\cup[1,\infty]$	η^{ρ}	ηt	$\eta^{\rho}t/\lambda$
Wiener	0	1	ηt	t/λ
Compound Poisson	$\rho \in (1,2)$	η^{ρ}	η^{ρ}	$\eta^{\rho}t/\lambda$
Gamma	2	η^{2}	ηt	$\eta^{2}t/\lambda$
IG	3	η^{3}	ηt	$\eta^{3}t/\lambda$

For the TED process, the PDF (1) of $Y(t)$ has no closed expression except for some special values [32]. According to the previous research [33-34], the saddle-point approximation (SAM) method provides a highly accurate approximation expression of PDF of $Y(t)$. Therefore, we adopt SAM to obtain the approximated PDF of TED process, which is expressed as

$$f(y \mid t,\rho,\eta,\lambda) \cong \sqrt{\frac{\lambda}{2\pi t^{1-\rho}y^{\rho}}}\cdot exp\{-\frac{\lambda t}{2}d(y \mid t,\eta)\}, \tag{3}$$

where $d(y \mid t,\eta)$ is called the unit deviance function, and which is expressed as

$$d(y \mid t,\eta) = \begin{cases} (y/t-\eta)^2, & \rho=0 \\ 2\left\{\frac{y}{t}ln\left(\frac{y}{\eta t}\right)-\left(\frac{y}{t}-\eta\right)\right\}, & \rho=1 \\ 2\{ln(\eta t/y)+y/(\eta t)-1\}, & \rho=2 \\ 2\left\{\frac{(max(y/t,0))^{2-\rho}}{(1-\rho)(2-\rho)}-\frac{\eta^{1-\rho}y}{(1-\rho)t}+\frac{\eta^{2-\rho}}{2-\rho}\right\}, & \rho\neq 0,1,2 \end{cases}. \tag{4}$$

2.2 Lifetime Distribution

The product's lifetime T is defined as the first passage time when the degradation path $Y(t)$ crosses a prespecified critical threshold ξ, that is

$$T = inf\{t \mid Y(t) \geq \xi\}. \tag{5}$$

According to Hong and Ye [35], the CDF of the lifetime T can be approached by the Birnbaum–Saunders distribution as

$$F_T(t) \cong \Phi\left[\sqrt{\frac{\lambda}{\eta^\rho}}\left(\eta\sqrt{t} - \frac{\xi}{\sqrt{t}}\right)\right], \tag{6}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Then, the PDF of the lifetime T is given by

$$f_T(t) = \frac{\eta t+\xi}{2t}\sqrt{\frac{\lambda}{2\pi\eta^\rho t}}exp\left[-\frac{\lambda}{2\eta^\rho}\left(\eta\sqrt{t} - \frac{\xi}{\sqrt{t}}\right)^2\right]. \tag{7}$$

2.3 Notations and assumptions

To use TED process for the purposes mentioned above, initial choices and assumptions have to be put forward. IT is assumed that:

(1) The upper stress bound is S_H, below which the failure mechanism is the same as the one under the normal use stress level S_0.

(2) The SSADT has γ stress levels and satisfies $S_0 \leq S_1 < S_2 < \cdots < S_\gamma \leq S_H$. At the beginning of the test, there are n units are selected and tested under stress S_1. After a period of time τ_1, the stress level is increased to S_2 until time τ_2. This continues until the stress increased to S_γ, and the test is finally terminated at time τ_γ. The stress level of SSADT can be expressed as:

$$S = \begin{cases} S_1, & 0 \leq t < \tau_1; \\ S_2, & \tau_1 \leq t < \tau_2; \\ \vdots & \vdots \\ S_\gamma, & \tau_{\gamma-1} \leq t < \tau_\gamma \end{cases}$$

(3) In the TED process, the parameter η denotes the degradation rate which obviously should be changed when the acceleration stress is different. The link function between degradation rate and stress level can follow one of the three functions below:

- Power law relation: $\eta(S_k) = \alpha S_k^\beta$;
- Arrhenius relation: $\eta(S_k) = \alpha exp(-\beta/S_k)$;
- Exponential relation: $\eta(S_k) = \alpha exp(\beta S_k)$.

The standardized stress levels are defined as

$$s_k = \begin{cases} (ln\, S_k - ln\, S_0)/(ln\, S_H - ln\, S_0), & \text{for the power law relation} \\ (1/S_0 - 1/S_k)/(1/S_0 - 1/S_H), & \text{for the Arrhenius relation} \\ (S_k - S_0)/(S_H - S_0), & \text{for the exponential relation} \end{cases} \tag{8}$$

Under the standardization, we can have $0 = s_0 \leq s_1 < s_2 < \cdots < s_\gamma \leq s_H = 1$, and we can obtain a unique form of the degradation rate:

$$\eta_k = \eta(s_k) = \alpha e^{\beta s_k} \tag{9}$$

(4) The test unit's degradation path under stress level s_k follows an TED process with unchanged parameters η_k, λ, ρ. That is

$$Y(t|s_k) \sim TED(\eta_k t, \lambda, \rho) \tag{10}$$

(5) The cumulative degradation function under the SSADT is derived as follows (For convenience, denote $\tau_0 = 0$):

$$Y(t) = \begin{cases} Y(t|s_1) \sim TED(\eta_1 t, \lambda, \rho), & 0 \le t < \tau_1 \\ Y(\tau_1|s_1) + Y\big((t-\tau_1)|s_2\big) \sim TED(\eta_2(t-\tau_1) + \eta_1\tau_1, \lambda, \rho), & \tau_1 \le t < \tau_2 \\ \vdots & \vdots \\ Y(\tau_1|s_1) + Y\big((\tau_2-\tau_1)|s_2\big) + \cdots + Y\left(\big(t-\tau_{\gamma-1}\big)|s_\gamma\right) \sim \\ \quad TED\big(\eta_\gamma\big(t-\tau_{\gamma-1}\big) + \sum_{i=1}^{\gamma-1}\eta_i\,(\tau_i - \tau_{i-1}), \lambda, \rho\big) & \tau_{\gamma-1} \le t < \tau_\gamma \end{cases} \tag{11}$$

The corresponding degradation path is shown in Fig. 1.

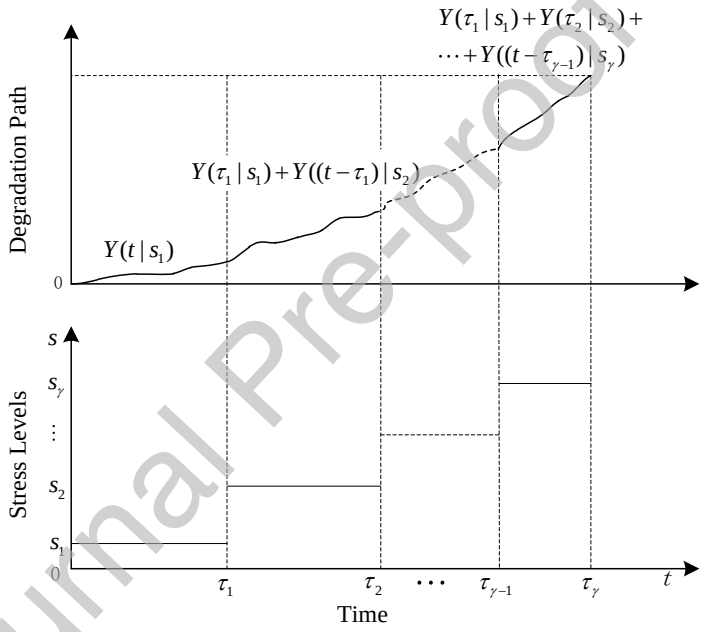

Fig. 1. The degradation path under SSADT

(6) Let m_k represent the number of inspections under s_k, and the inspections are conducted at the same inspection time interval Δt under all stress levels. We define the total number of inspections $M = \sum_{k=1}^{\gamma} m_k$. And set $p_k = m_k/M$ be the proportion of inspection number under stress level s_k.

2.4 Parameter estimation

Suppose that there are a total number of n test units subjected to the degradation test, and the *i*-th unit under the *j*-th stress level is measured at $t_{ijk} = k\Delta t$ with observations $Y_{ijk} = Y(t_{ijk})$, $1 \le i \le n$, $1 \le j \le m_k$, $k = 1,2,\cdots,\gamma$. Let $\Delta y_{ijk} = Y_{ijk} - Y_{ij(k-1)}$ be the observed degradation increments at the inspection interval $\Delta t = t_{ijk} - t_{ij(k-1)}$, then by the above assumptions

$$\Delta y_{ijk} \sim TED(\eta_k \Delta t, \lambda, \rho)\ . \tag{12}$$

Then the log-likelihood function is given by

$$l = \sum_{k=1}^{\gamma}\sum_{i=1}^{n}\sum_{j=1}^{m_k} ln f_{\Delta y}(\Delta y_{ijk} \mid \Theta)$$
$$= \sum_{k=1}^{\gamma}\sum_{i=1}^{n}\sum_{j=1}^{m_k}\left[\frac{ln(\lambda/2\pi)}{2} - \frac{(1-\rho)ln\Delta t}{2} - \frac{\rho}{2} ln(\Delta y_{ijk}) - \frac{\lambda\Delta t}{2} d(\Delta y_{ijk} \mid \Delta t, \eta_k)\right], \quad (13)$$

where $\boldsymbol{\Theta} = (\alpha, \beta, \lambda, \rho)$ is the parameter vector.

The maximum likelihood estimators (MLEs) of $\boldsymbol{\Theta}$ can be obtained by maximizing Eq.13. We may also take the first derivative of l with respect to λ. Setting this partial derivative to zero, then we can have

$$\hat{\lambda}_M = \frac{nM}{\Delta t \sum_{k=1}^{\gamma}\sum_{i=1}^{n}\sum_{j=1}^{m_k} d(\Delta y_{ijk}|\Delta t, \eta_k)}. \quad (14)$$

Then, the profile log-likelihood function of (α, β, ρ) can be given by substituting $\hat{\lambda}_M$ for λ in (13). Subsequently, the MLEs $(\hat{\alpha}_M, \hat{\beta}_M, \hat{\rho}_M)$ of (α, β, ρ) can be obtained by maximizing the profile log-likelihood function through a multiple-dimensional search. Here, we made use of the MATLAB function "fmincon" for this purpose. By substituting the MLEs $(\hat{\alpha}_M, \hat{\beta}_M, \hat{\rho}_M)$ of (α, β, ρ) into (14), then the MLE $\hat{\lambda}_M$ of λ can be obtained.

3. Optimal design of a SSADT

An SSADT is characterized by the total number of test units available *n*, the number γ of stress levels used in the test, as well as the stress value of each level, the allocation scheme of the measurements to each stress level, the test duration τ_γ, and the measurement time interval Δt. Here, we have assumed that the number of units, the test duration τ_γ, and the measurement time interval Δt are given. Therefore, in this section, the objective of the SSADT planning is to determine the optimal stress levels, as well as the proportion of units allocated to each level based on some optimization criterion. The procedure of optimal design is as follows:

Step 1: Derive the Fisher information matrix of the TED model, because many optimal SSADT design criteria are based on Fisher information matrix if the goal of conducting an experiment is to estimate the model parameters or their functions (e.g., lifetime percentiles). (see subsection 3.1).

Step 2: We briefly show the definitions of two commonly used optimality criteria: D-optimality and V-optimality. These two optimization criteria optimize the objective function from different perspectives. Sometimes one may be interested in different optimization criteria in different situations, or not only interested in a particular objective function.

Step 3: We then derive that for each criterion described in subsection 3.2, a multi-level SSADT plan, when optimized, degenerates to a simple SSADT plan using only the minimum and maximum stress levels. This result establishes a rationale for considering a simple SSADT using only the minimum and maximum stress levels.

Step 4: We consider the design problem of a simple SSADT plan and present the optimal allocation of inspections at each stress level for different criteria.

3.1 Fisher information matrix

The Fisher information matrix can be obtained by taking the expected values of the negative second derivatives of the log-likelihood function in Eq.13 with respect to the parameters. However, according to the research in [30-31,36-37], the Fisher information matrix could be singular and thereby the inverse matrix cannot be calculated because of the influence of the parameter ρ. Therefore, following to the research in [30-31,36-37], the effect of ρ on the Fisher information matrix does not be considered here, and which is replaced by $\hat{\rho}_M$. As a result, the parameter vector $\boldsymbol{\Theta}$ reduces to $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$. The Fisher information matrix $I(\boldsymbol{\theta})$ can be obtained by:

$$I(\boldsymbol{\theta}) = \begin{bmatrix} E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \beta}\right) & E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \lambda}\right) \\ E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \beta}\right) & E\left(-\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(-\frac{\partial^2 l}{\partial \beta\, \partial \lambda}\right) \\ E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \lambda}\right) & E\left(-\frac{\partial^2 l}{\partial \beta\, \partial \lambda}\right) & E\left(-\frac{\partial^2 l}{\partial \lambda^2}\right) \end{bmatrix}. \tag{15}$$

The elements of $I(\boldsymbol{\theta})$ are derived as follows:

$$\begin{cases} E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) &= n\lambda \sum_{k=1}^{\gamma} \sum_{j=1}^{m_k} \Delta t\, e^{2\beta s_k} \eta_k^{-\rho} \\ E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \beta}\right) &= n\alpha\lambda \sum_{k=1}^{\gamma} \sum_{j=1}^{m_k} \Delta t\, s_k e^{2\beta s_k} \eta_k^{-\rho} \\ E\left(-\frac{\partial^2 l}{\partial \alpha\, \partial \lambda}\right) &= 0 \\ E\left(-\frac{\partial^2 l}{\partial \beta^2}\right) &= n\alpha^2\lambda \sum_{k=1}^{\gamma} \sum_{j=1}^{m_k} \Delta t\, s_k^2 e^{2\beta s_k} \eta_k^{-\rho} \\ E\left(-\frac{\partial^2 l}{\partial \beta\, \partial \lambda}\right) &= 0 \\ E\left(-\frac{\partial^2 l}{\partial \lambda^2}\right) &= \frac{nM}{2\lambda^2} \end{cases}.$$

Let $A_k = e^{2\beta s_k} \eta_k^{-\rho} = \alpha^{-\rho} e^{(2-\rho)\beta s_k}$. Then, we have

$$I(\boldsymbol{\theta}) = \begin{bmatrix} nM\lambda\Delta t \sum_{k=1}^{\gamma} p_k A_k & nM\alpha\lambda\Delta t \sum_{k=1}^{\gamma} p_k s_k A_k & 0 \\ nM\alpha\lambda\Delta t \sum_{k=1}^{\gamma} p_k s_k A_k & nM\alpha^2\lambda\Delta t \sum_{k=1}^{\gamma} p_k s_k^2 A_k & 0 \\ 0 & 0 & \frac{nM}{2\lambda^2} \end{bmatrix} \tag{16}$$

3.2 Optimization criteria

Based on the above Fisher information matrix $I(\boldsymbol{\theta})$, the optimal stress levels, as well as the proportion of number of measurements allocated to each level can be determined subject to different optimality criteria. The D-optimality and V-optimality are two commonly used optimality criteria. Before proceeding further, we briefly review the definitions of these two commonly used criteria.

Criterion 1 (D-optimality). Actually, we need first to estimate the parameters of degradation model when we analyze the reliability. The accuracy of model parameter estimation will affect the accuracy of reliability. D-optimality focuses on the accuracy of parameters estimation and is based on

the maximization of the Fisher information matrix determinant. The principle of this criterion relies on the fact that the overall volume of the asymptotic joint confidence region of parameters is proportional to the inverse matrix of the Fisher information matrix. Motivated by this, maximizing the determinant of the Fisher information matrix is equivalent to minimizing asymptotic joint confidence ellipsoid of parameters, and then maximizing the joint precision of the estimators of parameters. The determinant of the Fisher information matrix can be obtained by:

$$det\big(I(\theta)\big) = \frac{1}{2}n^3\alpha^2\Delta t^2 M^3\left[\big(\sum_{k=1}^{\gamma} p_k\, A_k\big)\big(\sum_{k=1}^{\gamma} p_k A_k\, s_k^2\big) - \big(\sum_{k=1}^{\gamma} p_k\, A_k s_k\big)^2\right] \tag{17}$$

Criterion 2 (V-optimality). This optimality criterion is based on the minimization of the asymptotic variance of the estimated *p*-th percentile ($Avar\big(\hat{t}_p\big)$) of the lifetime distribution at common stress level s_0. The percentiles of product's lifetime distribution is an important reliability measures to be estimated. Using the Delta-method [10], the expression of $Avar\big(\hat{t}_p\big)$ is thereby given by

$$Avar\big(\hat{t}_p\big) = H(\theta)I^{-1}(\theta)H(\theta)^T \tag{18}$$

The p-th percentile of the lifetime distribution is derived as follows:

$$t_p = \frac{\left(Z_p\sqrt{\frac{\eta^\rho}{\lambda}} + \sqrt{Z_p^2\frac{\eta^\rho}{\lambda} + 4\eta\xi}\right)^2}{4\eta^2} \tag{19}$$

The elements of $H(\boldsymbol{\theta}) = \left(\frac{\partial t_p}{\partial \alpha}, \frac{\partial t_p}{\partial \beta}, \frac{\partial t_p}{\partial \lambda}\right)$ are derived as follows:

$$\begin{cases} \frac{\partial t_p}{\partial \alpha} = \frac{(h_1 + Z_p h_2)[(\rho-2)Z_p\eta^\rho(h_1 + Z_p h_2) - 4\lambda\eta\xi h_2]}{4\lambda\eta^3 h_1 h_2} \\ \frac{\partial t_p}{\partial \beta} = 0 \\ \frac{\partial t_p}{\partial \lambda} = -\frac{Z_p h_2 (h_1 + Z_p h_2)^2}{4\lambda\eta^2 h_1} \end{cases},$$

where $h_1 = \sqrt{\frac{Z_p^2\eta^\rho}{\lambda} + 4\eta\xi}$, $h_2 = \sqrt{\frac{\eta^\rho}{\lambda}}$, and Z_p is p-th percentile of the standard normal distribution.

3.3 Degeneration of an optimum multi-level SSADT plan

Theorem 1 Based on the D-optimality and V-optimality criteria above, if a test unit's degradation path follows the TED process and the relationship of drift parameter and stress satisfies Eq.9, the optimal multi-level SSADT plan using stress levels $s_1 < s_2 < \cdots < s_\gamma$ will degenerate to a simple SSADT plan using only the minimum and maximum stress levels, s_1 and s_γ.

Proof.

For **D-optimality**: For fixed total sample size n, Δt and M, maximization of $det\big(I(\theta)\big)$ in Eq.16 is equivalent to the maximization of a function *G* such as:

$$G\big(p_1, p_2, \cdots, p_\gamma\big) = \big(\sum_{k=1}^{\gamma} p_k\, A_k\big)\big(\sum_{k=1}^{\gamma} p_k\, A_k s_k^2\big) - \big(\sum_{k=1}^{\gamma} p_k\, A_k s_k\big)^2. \tag{20}$$

The further detailed proof can be found in Appendix A.

For **V-optimality**: The inverse of Fisher information matrix is

$$I^{-1}(\boldsymbol{\theta})=\begin{bmatrix} \frac{nM\alpha^2\lambda\Delta t\sum_{k=1}^{\gamma}p_k s_k^2 A_k}{n^2M^2\alpha^2\lambda^2\Delta t^2 G(p_1,p_2,\cdots,p_\gamma)} & -\frac{nM\alpha\lambda\Delta t\sum_{k=1}^{\gamma}p_k s_k A_k}{n^2M^2\alpha^2\lambda^2\Delta t^2 G(p_1,p_2,\cdots,p_\gamma)} & 0 \\ -\frac{nM\alpha\lambda\Delta t\sum_{k=1}^{\gamma}p_k s_k A_k}{n^2M^2\alpha^2\lambda^2\Delta t^2 G(p_1,p_2,\cdots,p_\gamma)} & \frac{nM\lambda\Delta t\sum_{k=1}^{\gamma}p_k A_k}{n^2M^2\alpha^2\lambda^2\Delta t^2 G(p_1,p_2,\cdots,p_\gamma)} & 0 \\ 0 & 0 & \frac{2\lambda^2}{nM} \end{bmatrix}.$$

Then, the asymptotic variance of the estimated p-th percentile of the failure time distribution is

$$Avar(\hat{t}_p)=\left(\frac{\partial t_p}{\partial\alpha}\right)^2\frac{(\sum_{k=1}^{\gamma}p_k s_k^2 A_k)}{n\lambda\Delta tM\left[(\sum_{k=1}^{\gamma}p_k A_k)(\sum_{k=1}^{\gamma}p_k s_k^2 A_k)-(\sum_{k=1}^{\gamma}p_k s_k A_k)^2\right]}+\frac{2\lambda^2}{nM}\left(\frac{\partial t_p}{\partial\lambda}\right)^2. \quad (21)$$

Hence, when n, Δt and M are fixed, minimizing this variance is equivalent to the minimization of

$$G_{Aver}=\frac{(\sum_{k=1}^{\gamma}p_k s_k^2 A_k)}{(\sum_{k=1}^{\gamma}p_k A_k)(\sum_{k=1}^{\gamma}p_k A_k s_k^2)-(\sum_{k=1}^{\gamma}p_k A_k s_k)^2}. \quad (22)$$

The further detailed proof can be found in Appendix A.

Notice that Theorem 1 provides an important insight regarding the optimum design of a SSADT plan. That is, the optimum SSADT plan is actually a simple SSADT using only minimum and maximum stress levels. Based on this observation, the next problem is how to allocate the inspection number under the two stress levels, which is shown in the following subsection 3.4.

3.4 Optimal simple SSADT plan

Since only the minimum and maximum stress levels are used in test, we can see that the decision variable is the inspection number m_1 under stress level s_1. Given the total inspection number M, we have $m_\gamma = M - m_1$ for stress level s_γ. The optimal allocation proportion based on each criterion is obtained as follows.

Theorem 2 For the D-optimality and V-optimality criteria, when the total sample size n, inspection number M and the time interval Δt between inspections are given, the optimum SSADT plan based on TED process is given as follows:

(1) For **D-optimality**: the optimum plan assigns inspections:

$$\left(\frac{m_1}{M},\frac{m_\gamma}{M}\right)=(p_1,p_\gamma)=\left(\frac{1}{2},\frac{1}{2}\right).$$

(2) For **V-optimality**: the optimum plan assigns inspections:

$$\left(\frac{m_1}{M},\frac{m_\gamma}{M}\right)=(p_1,p_\gamma)=\left(\frac{s_\gamma\sqrt{A_\gamma}}{s_1\sqrt{A_1}+s_\gamma\sqrt{A_\gamma}},\frac{s_1\sqrt{A_1}}{s_1\sqrt{A_1}+s_\gamma\sqrt{A_\gamma}}\right).$$

Proof. See Appendix B.

Based on Theorem 1 and Theorem 2, a simple SSADT plan uses the optimal allocation of inspections could generate the most efficient statistical results. For D-optimality, the optimum plan assigns inspections at maximum stress s_γ and minimum stress s_1 is equal, which is convenient for a degradation experiment. In other words, the optimum plan assigns inspections does not depend on the experimental design variables from a degradation test. For V-optimality, the optimum plan assigns

inspections is related to the difference between s_γ and s_1. It is worth mentioning that although the simple SSADT is optimum, a test may require more than two stress levels to verify whether the exponential relationship in (9) is valid. At times it could be beneficial to use more than two stress levels to attain more flexibility with some loss of efficiency.

4. A numerical example

This section illustrates the models and methods discussed above with a set of degradation data of LED chips [38]. Five units were placed at three temperature levels ($323K, 373K$ and $383K$) to conduct SSADTs. The experiment lasted for 13 months. During the experiment, the temperature of the first seven months was $323K$, the temperature of the eighth month to the eleventh month was $373K$, and the temperature of the twelfth month to the thirteenth month was $383K$. The percentage reduction in the light intensity of the five LEDs is recorded each month, and the data is shown in Table 2 and Fig.2. The normal operating temperature is set to $298K$, and the upper limit of the operating temperature is set to $383K$. When the light intensity of the led chip is reduced to 50% of the initial light intensity, the LED chip is considered to be fail. It can be seen from the data in Table 2 that the percentage of light intensity decreases. First, by subtracting the current value of the data in the table from 100%, the degraded data is converted to the increasing data suitable for describing the model in this paper, the failure threshold after transformation is $\xi = 50\%$.

Table 2. Percentage reduction in light intensity of LED chips at three temperature levels

Test Time (Month)	Temperature(K)	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5
1	323	97	98	96	99	95
2	323	96	97	94	95	93
3	323	92	94	93	93	88
4	323	91	92	90	91	85
5	323	88	89	85	85	83
6	323	86	86	82	84	82
7	323	85	82	80	79	77
8	373	82	80	77	77	75
9	373	81	79	72	73	73
10	373	77	76	70	71	68
11	373	75	74	68	69	65
12	383	71	70	65	63	60
13	383	62	60	58	59	55

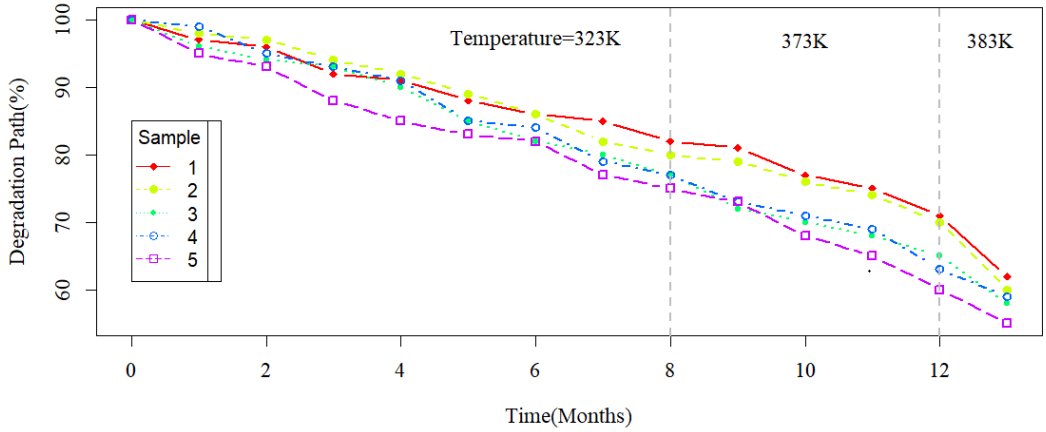

Fig.2 Degradation path of light intensity of LED chips under SSADT

4.1 Statistical inference

Under the assumption that the degradation process follows the TED process, the parameter Θ can be estimated as $\hat{\boldsymbol{\Theta}} = (\hat{\alpha}_M, \hat{\beta}_M, \hat{\lambda}_M, \hat{\rho}_M) = (2.25606, 0.52675, 4.023736, 2.18069)$. To present a reliable analysis, it is necessary to test whether the TED process model conforms to the observed degradation data. The Kolmogorov-Smirnov test is adopted. We select the degradation data as the samples at temperature 323K. Then, we propose the hypothesis to be tested, H_0: The data conforms to the theoretical distribution. The Kolmogorov-Smirnov test statistic is $D_{35} = 0.12198$ and the p-value is 0.631. The corresponding critical value with 0.05 level is $D_{35,0.05} = 0.224$. Obviously, $0.12198 < 0.224$, and the hypothesis H_0 cannot be rejected. The TED process fits the data well.

Furthermore, numerical comparisons of the log-likelihood function value and AIC value between the proposed TED model and the Wiener, Gamma, IG process models are presented in Table 3. The AIC value is calculated as AIC $= -2ln\hat{L} + 2N_{para}$, where $ln\hat{L}$ is the log-likelihood function value, and N_{para} is the number of model parameters. The model which has a larger log-likelihood function value and a smaller AIC value is preferable to describe data. From Table 3, it is can be seen that the proposed TED process model with $\hat{\rho}_M = 2.18069$ gets the maximal log-likelihood function value. Therefore, based on the log-likelihood function criterion, the TED process model with $\hat{\rho}_M = 2.18069$ can be considered as the most suitable degradation model for LED chip's degradation data. The log-likelihood function value obtained based on the Gamma process ($\rho = 2$) ranks second, and the smallest is that based on the Wiener process ($\rho = 0$). For the case where the power classification parameter ρ takes other values, the corresponding log-likelihood function values are shown in Figure 3. It can be seen that the farther the power classification parameter ρ is from the real value $\hat{\rho}_M$, the worse the model effect is. Therefore, the Wiener, Gamma and IG processes are suitable for describing the degradation data with power classification parameters of 0, 2, and 3, respectively. And if the real power classification parameter value ρ is far from 0, 2, and 3, then the fitting effect of these three

commonly used stochastic process models on degradation data will be poor. Considering the diversity of the products degradation process, these three stochastic process models have limitations, and the TED process has wider applicability. Taking the number of parameters into account, since the power classification parameter $\hat{\rho}_M = 2.18069$ is very close to 2, which correspond to Gamma process, and the TED process model has one more parameter, the Gamma process model has the minimum AIC value, followed by the TED process, and that based on Wiener process is largest. The farther the power classification parameter ρ is from the real value $\hat{\rho}_M$, the larger the AIC value. Considering that the AIC value of the TED process is very close to that of Gamma process, and TED process has largest log-likelihood function value. In addition, the power classification parameter $\hat{\rho}_M = 2.18069$, but the power classification parameter of Gamma process is 2, which means that the Gamma process is an approximation to the true degenerate process. The TED process is more accurate in describing the real degradation process. Although the TED process is more complicated than Gamma process, there is only one more parameter, which is not too complicated and can be calculated effectively. Therefore, although Gamma process is also a good choice for this data from AIC, considering flexibility and wide range of applications, the TED process is used to characterize this set of data.

Table 3 Comparison the TED process with three well-known processes

	ρ	α	β	λ	Log-likelihood	AIC
TED	2.18069	2.25606	0.52675	4.023736	-117.2488	242.4976
Wiener	0	2.216247	0.5565	0.3221178	-129.0482	264.0964
Gamma	2	2.249493	0.53148	3.355369	-117.6303	241.2606
IG	3	2.282095	0.50746	8.609644	-118.9262	243.8524

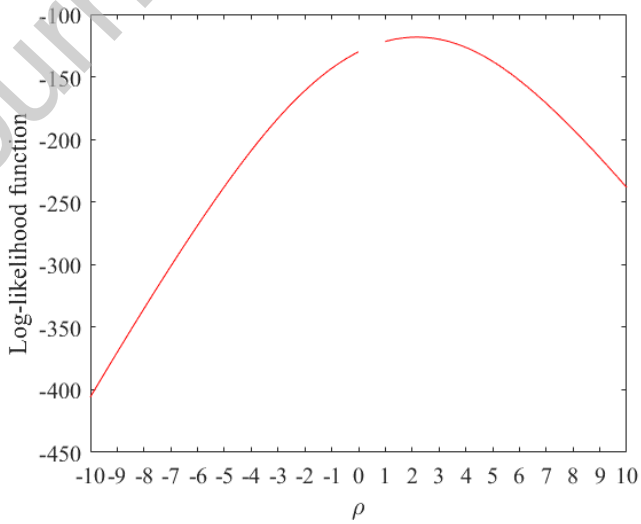

Fig.3. The log-likelihood functions based on different power classification parameter ρ

4.2 Optimum SSADT plans

It is supposed that a SSADT have 5 LED chips for testing, the previous three stress levels (0.3487525, 0.9060085 and 1.0000000) are to be used when conducting a SSADT and there are 13 inspections to be allocated into each stress level. Therefore, the total sample size $n = 5$, inspection number $M = 13$ and the inspection interval $\Delta t = 1$. According to Theorem 2 in subsection 3.4, the optimal allocations of inspections are calculated by multiplying the theoretical optimal proportions by 13. For D-optimality, we have (m_1',m_3') = (6.5,6.5); for V-optimality, we have (m_1',m_3') = (9.560719, 3.439281). It is can be seen these numbers are decimals, thus the nearest integers are used. For D-optimality, (6, 7) or (7, 6) inspections would be allocated at stress levels s_1 and s_3, respectively; for V-optimality, (9, 4) or (10, 3) inspections would be allocated at stress levels s_1 and s_3 respectively.

To compare any given SSADT plans with the above optimal simple SSADT plan, we compute the values of the objective function for each SSADT plan, i.e., the determinant of the Fisher information matrix and asymptotic variance of the estimated p-th percentile. In addition, the Relative Efficiency (RE) of a SSADT plan are also calculated, which are defined as

$$RE(m_1, m_2, m_3) = \frac{det(I)\text{corresponding to the SSADT plan } (m_1, m_2, m_3)}{det(I)\text{corresponding to the optimal SSADT plan for D-optimality}}$$

and

$$RE(m_1, m_2, m_3) = \frac{Avar(\hat{t}_{0.1}) \text{ corresponding to the optimal SSADT plan for V-optimality}}{Avar(\hat{t}_{0.1}) \text{ corresponding to the SSADT plan } (m_1, m_2, m_3)}$$

The results of comparisons between the optimal SSADT plans and the original SSADT plan are presented in Table 4. For D-optimality criteria, the optimum plan could improve the efficiency by 18 percent according to the determinant of the Fisher information matrix $Det(I)$. However, the optimum plan $(7,0,6)$ has lower the efficiency than the original plan by 2.88 percent according to the $Avar(\hat{t}_{0.1})$. For V-optimality criteria, on the one hand, the optimum plan could improve the efficiency about 19.73 percent according to the $Avar(\hat{t}_{0.1})$. On the other hand, it can be found that the optimum plan (10,0,3) has lower the efficiency than the original plan by 11 percent according to the $Det(I)$. According to the results of Table 4, no matter which kind of optimality criteria, the result of the plans which are used by rounding the theoretical allocation of measurements to nearest integers is almost equal. And for product's reliability indicators of interest, the experimenter can significantly improve efficiency by using the optimal number of measurements under the corresponding optimization criteria.

Table 4: Comparisons between the original SSADT plan and the optimal plans

	(m_1,m_2,m_3)	$Det(I)$(RE)	$Avar(\hat{t}_{0.1})$(RE)
Original plan	(7, 4, 2)	1546.279 (0.8296)	102.7117 (0.8027)
D-optimality	(7, 0, 6)	1863.882 (1.0000)	106.5393 (0.7739)
	(6, 0, 7)	1863.882 (1.0000)	94.6537 (0.8710)
V-optimality	(9, 0, 4)	1597.613 (0.8571)	82.6341 (0.9977)
	(10, 0, 3)	1331.344 (0.7143)	82.4465 (1.0000)

5. Simulation study

In this section, to demonstrate that the SSADT designs presented above also have good performance than other SSADT plans with different sample size, we conducted a simulation study here. The following experimental settings are considered: the number of samples $N = 3, 20, 35$ and 50, $\gamma = 4$, (s_1, s_2, s_3, s_4) = $(1/4, 2/4, 3/4, 1)$, $(\alpha,\beta,\rho,\lambda)$ = (2, 2.5, 1.8, 50), and the total number of measurements $M = 12, 24$ and 36. The optimal allocations of measurements are calculated according to Theorem 2. If the theoretical allocation is not an integer, the nearest integers are used as above. And an equally allocated SSADT plan is compared with the result of each optimum SSADT plan. The simulation results are presented in Tables 5 and 6, respectively. The following observations can be drawn from the simulation results:

(1) For the D-optimality criteria, with the increase of sample size N or the total measurement number M, the determinant of Fisher information matrix increases for all SSADT plans. And the differences of determinant values for different SSADT plan increase as N or M increase. When the sample size N or the measurement number M are relatively small ($N \leq 20$, or $M \leq 24$), The determinant value grows relatively slowly, and the difference between the determinant values is relatively small; When N or M is relatively large ($N > 20$,or $M > 24$), the determinant value growth is relatively fast, and the difference between the determinant values is relatively large. Regardless of how N or M change, the determinant value of the optimum plan is the largest. Therefore, the D-optimality criteria is sensitive to the accelerated degradation test plan, the sample size N and the measurement number M.

In terms of RE, the optimum plans under D-optimality have significantly higher efficiency than do other plans, the optimum plans under V-optimality have minimal RE. Therefore, the optimum plan can significantly improve the efficiency, and the D-optimality criteria is relatively sensitive to the plan used. Furthermore, the RE is less affected by the sample size N and the measurement number M, which means the RE under D-optimality is robust to the sample size N and the measurement number M. And the optimum plan under D-optimality performance well in small sample size.

(2) For the V-optimality criteria, with the increase of sample size N or the total measurement number M, the asymptotic variance of the life quantile decreases for all SSADT plans. When the sample size N or the measurement number M is relatively small ($N \leq 20$,or $M \leq 24$), the asymptotic variance for all SSADT plans descend faster, and the difference between the asymptotic variances is relatively large; When the sample size N or the measurement number M is relatively large ($N > 20$, or $M > 24$), the asymptotic variances for all SSADT plans decrease slowly, and the difference between the asymptotic variances is relatively small. That is, too small sample size N or the measurement number M will cause large asymptotic variance of the life quantile, large sample size N or the measurement number M will not cause a large reduction in variance. These situations provide a reference for us to choose an appropriate sample size N or the measurement number M. In addition, regardless of how N or M change, the asymptotic variance of the optimum plan under V-optimality is the smallest. The plan that assigns the same inspection number under the four stress levels has the highest variance. Therefore, when the sample size is relatively small, the V-optimality criteria is relatively sensitive to the plan used, we need to choose the accelerated degradation test plan carefully, and the optimum plan is recommended to be adopted; When the sample size is relatively large, the overall asymptotic variance is relatively small, the V-optimality criteria is relatively robust to the plan used.

In terms of RE, the optimum plan under V-optimality performs best and nearly full efficiency, the plan that assigns the same inspection number for the given four stress levels has the smallest RE. Therefore, the optimum plan can significantly improve the efficiency. In addition, the RE is less affected by the sample size N and the measurement number M, which means the RE under V-optimality is robust to the sample size N and the measurement number M.

Table 5: Theoretical results for each SSADT plan

		$N = 3$		$N = 20$		$N = 35$		$N = 50$	
M	(m_1, m_2, m_3, m_4)	$Det(I)$(RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)
12	(3,3,3,3)	1130.23 (0.5590)	1.5268 (0.4718)	334884.084 (0.5590)	0.2290 (0.4718)	1794769.389 (0.5590)	0.1309 (0.4718)	5232563.816 (0.5590)	0.0916 (0.4718)
	(6,0,0,6)	2021.74 (1.000)	1.0285 (0.7004)	599035.173 (1.000)	0.1543 (0.7004)	3210454.129 (1.000)	0.0882 (0.7004)	9359924.575 (1.000)	0.0617 (0.7004)
	(10,0,0,2)	1123.19 (0.5556)	0.7204 (1.000)	332797.318 (0.5556)	0.1081 (1.000)	1783585.627 (0.5556)	0.0617 (1.000)	5199958.097 (0.5556)	0.0432 (1.000)
	(9,0,0,3)	1516.31 (0.7500)	0.7436 (0.9687)	449276.38 (0.7500)	0.1115 (0.9687)	2407840.597 (0.7500)	0.0637 (0.9687)	7019943.431 (0.7500)	0.0446 (0.9687)
24	(6,6,6,6)	9041.870 (0.5590)	0.7634 (0.4718)	2679072.67 (0.5590)	0.1145 (0.4718)	14358155.11 (0.5590)	0.0654 (0.4718)	41860510.53 (0.5590)	0.0458 (0.4718)
	(12,0,0,12)	16173.950 (1.000)	0.5142 (0.7004)	4792281.38 (1.000)	0.0771 (0.7004)	25683633.03 (1.000)	0.0441 (0.7004)	74879396.6 (1.000)	0.0308 (0.7004)
	(20,0,0,4)	8985.53 (0.5556)	0.3602 (1.000)	2662378.55 (0.5556)	0.0540 (1.000)	14268685.02 (0.5556)	0.0309 (1.000)	41599664.78 (0.5556)	0.0216 (1.000)
	(19,0,0,5)	10670.31 (0.6597)	0.3630 (0.9922)	3161574.52 (0.6597)	0.0544 (0.9922)	16944063.46 (0.6597)	0.0311 (0.9922)	49399601.92 (0.6597)	0.0217 (0.9922)
36	(9,9,9,9)	30516.31 (0.5590)	0.5089 (0.4718)	9041870.27 (0.5590)	0.0763 (0.4718)	48458773.5 (0.5590)	0.0436 (0.4718)	141279223 (0.5590)	0.0305 (0.4718)
	(18,0,0,18)	54587.08 (1.000)	0.3428 (0.7004)	16173949.7 (1.000)	0.0514 (0.7004)	86682261.49 (1.000)	0.0294 (0.7004)	252717963.5 (1.000)	0.0205 (0.7004)
	(30,0,0,6)	30326.16 (0.5556)	0.2401 (1.000)	8985527.59 (0.5556)	0.0360 (1.000)	48156811.94 (0.5556)	0.0206 (1.000)	140398868.6 (0.5556)	0.0144 (1.000)
	(29,0,0,7)	34201.16 (0.6265)	0.2409 (0.9969)	10133678.3 (0.6265)	0.0361 (0.9969)	54310182.35 (0.6265)	0.0206 (0.9969)	158338724.1 (0.6265)	0.0144 (0.9969)

Table 6: The average of 1000 simulated results for each SSADT plan

		$N = 3$		$N = 20$		$N = 35$		$N = 50$	
M	(m_1, m_2, m_3, m_4)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)	$Det(I)$ (RE)	Avar($t_{0.1}$) (RE)
12	(3,3,3,3)	1.188203e+03 (0.5942)	0.6477 (0.7296)	3.240802e+05 (0.5412)	0.2132 (0.4866)	1.777996e+06 (0.5455)	0.1267 (0.4767)	5.221907e+06 (0.5636)	0.0894 (0.4765)
	(6,0,0,6)	1.999628e+03 (1.000)	0.6242 (0.7570)	5.987476e+05 (1.0000)	0.1472 (0.7047)	3.259160e+06 (1.0000)	0.0865 (0.6978)	9.265883e+06 (1.0000)	0.0607 (0.7016)
	(10,0,0,2)	1.153975e+03 (0.5771)	0.4725 (1.0000)	3.327246e+05 (0.5557)	0.1037 (1.0000)	1.785328e+06 (0.5478)	0.0604 (1.0000)	5.162781e+06 (0.5572)	0.0426 (1.0000)
	(9,0,0,3)	1.555858e+03 (0.7781)	0.4955 (0.9537)	4.616349e+05 (0.7710)	0.1076 (0.9637)	2.394626e+06 (0.7347)	0.0625 (0.9653)	7.094643e+06 (0.7657)	0.0441 (0.9653)
24	(6,6,6,6)	9.221750e+03 (0.5858)	0.5118 (0.5646)	2.596628e+06 (0.5397)	0.1104 (0.4787)	1.428755e+07 (0.5518)	0.0645 (0.4732)	4.145059e+07 (0.5540)	0.0452 (0.4762)
	(12,0,0,12)	1.574240e+04 (1.0000)	0.4030 (0.7170)	4.811627e+06 (1.0000)	0.0755 (0.7001)	2.589298e+07 (1.0000)	0.0437 (0.6982)	7.481956e+07 (1.0000)	0.0307 (0.7015)
	(20,0,0,4)	9.753330e+03 (0.6196)	0.2889 (1.0000)	2.682511e+06 (0.5575)	0.0528 (1.0000)	1.419131e+07 (0.5481)	0.0305 (1.0000)	4.173539e+07 (0.5578)	0.0215 (1.0000)
	(19,0,0,5)	1.079573e+04 (0.6858)	0.2942 (0.9820)	3.179363e+06 (0.6608)	0.0534 (0.9902)	1.677987e+07 (0.6480)	0.0308 (0.9920)	4.998070e+07 (0.6680)	0.0217 (0.9926)
36	(9,9,9,9)	2.981906e+04 (0.5479)	0.3872 (0.5400)	8.847249e+06 (0.5454)	0.0744 (0.4771)	4.832393e+07 (0.5580)	0.0432 (0.4745)	1.423945e+08 (0.5666)	0.0303 (0.4730)
	(18,0,0,18)	5.442127e+04 (1.0000)	0.2920 (0.7162)	1.622179e+07 (1.0000)	0.0507 (0.7000)	8.660255e+07 (1.0000)	0.0292 (0.7016)	2.513005e+08 (1.0000)	0.0205 (0.7005)
	(30,0,0,6)	3.186558e+04 (0.5855)	0.2091 (1.0000)	8.791722e+06 (0.5420)	0.0355 (1.0000)	4.842867e+07 (0.5592)	0.0205 (1.0000)	1.404175e+08 (0.5588)	0.0143 (1.0000)
	(29,0,0,7)	3.406215e+04 (0.6259)	0.2093 (0.9989)	1.013832e+07 (0.6250)	0.0358 (0.9922)	5.433593e+07 (0.6274)	0.0205 (0.9976)	1.588978e+08 (0.6323)	0.0144 (0.9954)

6. Conclusions

In this paper, we investigated the optimal SSADT plan problem for the TED process based on the D-optimality criterion and V-optimality criterion. Although many research efforts have been devoted to finding the optimum SSADT plan, most of the results are based primarily on numerical studies or special degradation models. In this paper, the studied TED process is a general class of degradation model, which includes some commonly used stochastic process as its special cases, e.g., Wiener, Gamma, and IG processes. In addition, we have derived that the optimal SSADT plan with multiple stress levels based on TED process with an exponential drift parameter-stress relationship in a formal manner. Our results suggest that the optimal SSADT actually is the simple SSADT with two stress levels, that is the minimum stress level s_1 and the maximum stress level s_γ. Furthermore, we have theoretically derived the optimal allocation of inspections at each stress level based on D-optimality and V-optimality criteria, respectively.

A numerical example, a study of SSADT of LEDs' light intensity, is provided for comparison. The results indicate that the efficiency is improved by using the optimum simple SSADT plan. Furthermore, the Monte Carlo simulations are used to study the performances of the optimum plan and some non-optimum plans. The results also show that the efficiency is improved by using the optimum simple SSADT plan.

In the future work, when the parameter-stress relationship is not an exponential function, the optimal SSADT design can be considered. Also, it will be useful to search for a robust SSADT plan in the sense that the plan could provide reasonable efficiencies to many objectives of interest simultaneously.

CRediT authorship contribution statement

Weian Yan: Conceptualization, Methodology, Software, Formal analysis, Writing – original draft. **Xiaofan Xu:** Methodology, Software, Writing – review & editing. **David Bigaud**: Supervision. **Wenqin Cao**: Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We would like to thank the editor and three reviewers for careful work and remarkable comments which have led to substantial improvements to an early version of the paper. This work is support by the National Natural Science Foundation of China (no.71861011).

Appendix A

PROOF OF THEOREM 1.

For **D-optimality**: For fixed total sample size n, Δt and M, maximizing $det\big(I(\theta)\big)$ in Eq.17 is equivalent to maximizing

$$G(p_1, p_2, \cdots, p_\gamma) = \big(\sum_{k=1}^{\gamma} p_k A_k\big)\big(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\big) - \big(\sum_{k=1}^{\gamma} p_k s_k A_k\big)^2$$

Let

$g_1(s_k) = A_k = \alpha^{-\rho} e^{(2-\rho)\beta s_k}$,

$g_2(s_k) = s_k^2 A_k = s_k^2 \alpha^{-\rho} e^{(2-\rho)\beta s_k}$ and

$g_3(s_k) = s_k A_k = s_k \alpha^{-\rho} e^{(2-\rho)\beta s_k}$.

Then

$\frac{dg_1}{ds_k} = \alpha^{-\rho}(2-\rho)\beta e^{(2-\rho)\beta s_k}$,

$\frac{dg_2}{ds_k} = (2+(2-\rho)s_k\beta) s_k \alpha^{-\rho} e^{(2-\rho)\beta s_k}$, and

$\frac{dg_3}{ds_k} = \alpha^{-\rho}(1+(2-\rho)\beta s_k) e^{(2-\rho)\beta s_k}$.

Therefore,

$\frac{dg_1}{ds_k} = 0$,when$\rho = 2$ $\quad \frac{dg_1}{ds_k} > 0$,when$\rho < 2$ $\quad \frac{dg_1}{ds_k} < 0$,when$\rho > 2$

$\frac{dg_2}{ds_k} = 0$,when$\rho = 2 + \frac{2}{\beta s_k}$ $\quad \frac{dg_2}{ds_k} > 0$,when$\rho < 2 + \frac{2}{\beta s_k}$ $\quad \frac{dg_2}{ds_k} < 0$,when$\rho > 2 + \frac{2}{\beta s_k}$

$\frac{dg_3}{ds_k} = 0$,when$\rho = 2 + \frac{1}{\beta s_k}$ $\quad \frac{dg_3}{ds_k} > 0$,when$\rho < 2 + \frac{1}{\beta s_k}$ $\quad \frac{dg_3}{ds_k} < 0$,when$\rho > 2 + \frac{1}{\beta s_k}$

When $\rho \le 2$, $g_1(s_k), g_2(s_k)$ and $g_3(s_k)$ is monotone increasing. Then,

$$\big(\sum_{k=1}^{\gamma} p_k A_k\big)\big(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\big) - \big(\sum_{k=1}^{\gamma} p_k s_k A_k\big)^2 \le \big(p_1 A_1 + (1-p_1)A_\gamma\big)\big(p_1 s_1^2 A_1 + (1-p_1)s_\gamma^2 A_\gamma\big) - \Big(\big((1-p_\gamma)s_1 A_1 + p_\gamma s_\gamma A_\gamma\big)\Big)^2$$

When $\rho > 2 + \frac{2}{\beta s_k}$, $g_1(s_k), g_2(s_k)$ and $g_3(s_k)$ is monotone decreasing. Then,

$$\big(\sum_{k=1}^{\gamma} p_k A_k\big)\big(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\big) - \big(\sum_{k=1}^{\gamma} p_k s_k A_k\big)^2 \le \Big(\big((1-p_\gamma)A_1 + p_\gamma A_\gamma\big)\Big)\Big(\big((1-p_\gamma)s_1^2 A_1 + p_\gamma s_\gamma^2 A_\gamma\big)\Big) - \big(p_1 s_1 A_1 + (1-p_1)s_\gamma A_\gamma\big)^2$$

When $2 < \rho \le 2 + \frac{1}{\beta s_k}$, $g_1(s_k)$ is monotone decreasing, $g_2(s_k)$ and $g_3(s_k)$ is monotone increasing. Then,

$$\left(\sum_{k=1}^{\gamma} p_k A_k\right)\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right) - \left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2 \leq \left(p_1 A_1 + (1-p_1)A_\gamma\right)\left(p_1 s_1^2 A_1 + (1-p_1)s_\gamma^2 A_\gamma\right) - \left((1-p_\gamma)s_1 A_1 + p_\gamma s_\gamma A_\gamma\right)^2$$

When $2 + \frac{1}{\beta s_k} < \rho \leq 2 + \frac{2}{\beta s_k}$, $g_1(s_k)$ and $g_2(s_k)$ is monotone decreasing, $g_3(s_k)$ is monotone increasing. Then,

$$\left(\sum_{k=1}^{\gamma} p_k A_k\right)\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right) - \left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2 \leq \left((1-p_\gamma)A_1 + p_\gamma A_\gamma\right)\left(p_1 s_1^2 A_1 + (1-p_1)s_\gamma^2 A_\gamma\right) - \left(p_1 s_1 A_1 + (1-p_1)s_\gamma A_\gamma\right)^2$$

For **V-optimality**: For fixed total sample size n, Δt and M, minimizing Eq.22 is equivalent to minimizing

$$G_{Aver} = \frac{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}{\left(\sum_{k=1}^{\gamma} p_k A_k\right)\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right) - \left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2} = 1/\left[\left(\sum_{k=1}^{\gamma} p_k A_k\right) - \frac{\left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2}{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}\right].$$

When $\rho \leq 2$, $g_1(s_k), g_2(s_k)$ and $g_3(s_k)$ is monotone increasing. Then,

$$1/\left[\left(\sum_{k=1}^{\gamma} p_k A_k\right) - \frac{\left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2}{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}\right] \geq 1/\left[\left(p_1 A_1 + (1-p_1)A_\gamma\right) - \frac{\left((1-p_\gamma)s_1 A_1 + p_\gamma s_\gamma A_\gamma\right)^2}{(1-p_\gamma)s_1^2 A_1 + p_\gamma s_\gamma^2 A_\gamma}\right]$$

When $\rho > 2 + \frac{2}{\beta s_k}$, $g_1(s_k), g_2(s_k)$ and $g_3(s_k)$ is monotone decreasing. Then,

$$1/\left[\left(\sum_{k=1}^{\gamma} p_k A_k\right) - \frac{\left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2}{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}\right] \geq 1/\left[\left((1-p_\gamma)A_1 + p_\gamma A_\gamma\right) - \frac{\left(p_1 s_1 A_1 + (1-p_1)s_\gamma A_\gamma\right)^2}{p_1 s_1^2 A_1 + (1-p_1)s_\gamma^2 A_\gamma}\right]$$

When $2 < \rho \leq 2 + \frac{1}{\beta s_k}$, $g_1(s_k)$ is monotone decreasing, $g_2(s_k)$ and $g_3(s_k)$ is monotone increasing. Then,

$$1/\left[\left(\sum_{k=1}^{\gamma} p_k A_k\right) - \frac{\left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2}{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}\right] \geq 1/\left[\left((1-p_\gamma)A_1 + p_\gamma A_\gamma\right) - \frac{\left((1-p_\gamma)s_1 A_1 + p_\gamma s_\gamma A_\gamma\right)^2}{(1-p_\gamma)s_1^2 A_1 + p_\gamma s_\gamma^2 A_\gamma}\right]$$

When $2 + \frac{1}{\beta s_k} < \rho \leq 2 + \frac{2}{\beta s_k}$, $g_1(s_k)$ and $g_2(s_k)$ is monotone decreasing, $g_3(s_k)$ is monotone increasing. Then,

$$1/\left[\left(\sum_{k=1}^{\gamma} p_k A_k\right) - \frac{\left(\sum_{k=1}^{\gamma} p_k s_k A_k\right)^2}{\left(\sum_{k=1}^{\gamma} p_k s_k^2 A_k\right)}\right] \geq 1/\left[\left((1-p_\gamma)A_1 + p_\gamma A_\gamma\right) - \frac{\left((1-p_\gamma)s_1 A_1 + p_\gamma s_\gamma A_\gamma\right)^2}{p_1 s_1^2 A_1 + (1-p_1)s_\gamma^2 A_\gamma}\right]$$

Appendix B

PROOF OF THEOREM 2

For **D-optimality**: we define:

$$h_{det}(p_1, p_\gamma) = G(p_1, p_\gamma) + B(1 - p_1 - p_\gamma)$$

Taking the derivative with respect to p_1 and p_γ, we have : $\frac{\partial h_{det}(p_1,p_\gamma)}{\partial p_1}$ and $\frac{\partial h_{det}(p_1,p_\gamma)}{\partial p_\gamma}$. With solving the above two equations for B and setting the two expression equal to each other , we can find the optimal values as $p_1^* = p_\gamma^* = \frac{1}{2}$.

For **V-optimality**: we define:

$$h_{Aver}(p_1, p_\gamma) = G_{Aver}(p_1, p_\gamma) + B(1 - p_1 - p_\gamma)$$

Taking the derivative with respect to p1 and pm, we have: $\frac{\partial h_{Aver}(p_1,p_\gamma)}{\partial p_1}$ and $\frac{\partial h_{Aver}(p_1,p_\gamma)}{\partial p_\gamma}$. With solving the above two equations for B and setting these two expressions equal to each other, we can find the optimal values as $\frac{p_1^2}{p_\gamma^2} = \frac{s_\gamma^2 A_\gamma}{s_1^2 A_1}$.

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