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► **To cite this version:**

Florian Navarro. Associated consistency and the Aumann-Drèze value. *International Game Theory Review*, In press, 10.1142/S0219198922500189 . hal-03678064

HAL Id: hal-03678064

<https://univ-angers.hal.science/hal-03678064>

Submitted on 25 May 2022

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Associated consistency and the Aumann-Drèze value

Florian NAVARRO

Received: date / Accepted: date

Abstract This article presents an axiomatic characterization of the Aumann-Drèze value (1974) for cooperative games with coalition structures. We build an associated game that extends the original associated game presented by Hamiache (2001) to cooperative games with coalition structures. We use a similar approach to the one used in Hamiache and Navarro (2020). This new associated game is expressed through a matrix form. We show that the series of successive associated games is convergent and that its limit is an inessential game. This allows us to propose a characterization of the Aumann-Drèze value that relies on associated consistency, inessential game and continuity axioms. Hence, this paper strengthens the results of Hamiache (2001) and Hamiache and Navarro (2020) considering that if these axioms are viewed as desirable, we are now able to provide a unique value for three different types of problems : the Shapley value on standard games (Hamiache 2001), the Hamiache-Navarro value on games with graphs (Hamiache and Navarro 2020) and the Aumann-Drèze value for games with coalition structures.

Keywords game theory, cooperative games, coalition structures, associated consistency, shapley value, aumann-drèze value

1 Introduction

This article deals with cooperative games with coalition structures as introduced in Aumann and Drèze [1]. This framework extended the standard model of cooperative games to allow a restriction of communication, and thus cooperation, between players. Players are partitioned into subsets of the player set called blocks. Cooperation is complete inside blocks and impossible between

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blocks. On this framework, the authors developed a generalization of the Shapley value (Shapley, [13]), which is known as the Aumann-Drèze value. To the best of our knowledge, the characterization offered in van den Nouweland and Slikker [12] is the only other axiomatic result on the Aumann-Drèze value that differs from the original one. In this article we propose a new characterization of this value.

Our work is in line with the recent literature on associated consistency as introduced by Hamiache [7]. It joins a family of work offering axiomatic characterizations of sharing rules (for different type of games) that rely on an associated consistency axiom. This axiom demands that a sharing rule gives the same pay-off to the game or a perturbation of the original game called an associated game. This invariance principle ensures that the sharing rule is robust to a certain type of manipulation that coalitions could make on the evaluation of their own worth.

Beside its seminal use by Hamiache to characterize the Shapley value (Hamiache [7], [5]) it has been used in Hwang [9] to characterize the equal allocation of non-separable costs, in Driessen [2] to offer a characterization of the family of values of symmetric, linear and efficient sharing rules and by Xu *et al.* [14] for the Equal Surplus Division value. For games with graph (as introduced in Myerson [10]), the associated consistency axiom proved effective for finding new sharing rules. We can cite the F-Value (Hamiache [8]), the Mean-Value (Hamiache [6]), the Center-Value (Navarro [11]) as well as the Hamiache-Navarro value (Hamiache and Navarro [3]).

In Hamiache [4], the author presents a characterization of an efficient extension of the Aumann-Drèze value as well as a family of value that are convex combination of the Shapley value and the Aumann-Drèze value and relies also on associated games and their matrix form. However the associated games are constructed using similarity matrices and are profoundly different to the associated games developed in this article.

This present work is similar to the work done in Hamiache and Navarro [3] in that it also extends the characterization of the Shapley value offered in Hamiache [7] to situations of restricted communication. In Hamiache and Navarro [3] the authors adapted the axioms used in Hamiache [7] to the framework of games with graphs. The associated game was also modified so that instead of considering the coalition $N \setminus S$, coalition S only considered its immediate neighbours. This paper attempts to do the same extension of Hamiache [7] to the framework of games with coalition structure. The axioms used in this article are direct adaptations of the aforementioned ones to this framework. The associated game is modified such that a coalition S only considers the members of its own block. Alongside continuity and an inessential game axioms, we then use the associated game axiom to offer a new characterization for the Aumann-Drèze Value.

It shall be noted that, since games with coalition structures can be expressed as particular cases of games with graphs (see Nouweland and Slikker [12]), some of the results proposed in this article can be obtained as corollaries of results presented in Hamiache and Navarro [3]. However, these results are

insufficient on their own to pinpoint the Aumann-Drèze value. Moreover, this article develops new proofs specific to the framework of coalition structures that are considerably simpler, in part due to a specific order on the set of coalitions.

In the next section we present the general framework. In section 3 we present the associated game and its matrix form $M_{\mathcal{B}}$ as well as several results and properties of this matrix. In section 4 we introduce our system of axioms and present our main characterization result.

2 Formal framework

Let U be a non-empty and finite set of players. A coalition is a non-empty subset of U . A coalitional game with transferable utility (a TU-game) is a pair (N, v) where N is a coalition and v is a function satisfying $v : 2^N \rightarrow \mathbb{R}$ and $v(\emptyset) = 0$. We note Γ the set of these games. Given a coalition $S \subseteq N$ we write s its cardinality. For any $S \subsetneq N$ we will write $(S, v|_S)$ the sub-game restricted to coalition S with $v|_S(T) = v(T)$, $\forall T \subseteq S$. A game (N, v) is said to be inessential if for all pairs of disjoint coalitions $S \subseteq N$ and $T \subseteq N \setminus S$, $v(S \cup T) = v(S) + v(T)$. Note that a game (N, v) is inessential if and only if, for all coalitions $S \subseteq N$, $v(S) = \sum_{i \in S} v(\{i\})$. A unanimity game (N, u_R) is defined such that $u_R(S) = 1$ if $R \subseteq S$, and 0 otherwise.

In this article the restriction of cooperation is represented by coalition structure. Given a set of players N , a coalition structure is a couple (N, \mathcal{B}) with \mathcal{B} a partition of N such that $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ with $\bigcup_{1 \leq \alpha \leq m} B_\alpha = N$ and $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$ for any $\alpha_1, \alpha_2 \in \{1, 2, \dots, m\}$ such that $\alpha_1 \neq \alpha_2$. The sets B_α are called blocks. For any coalition $S \subseteq N$ we shall note $B(S) = \{B \cap S \mid B \in \mathcal{B}, B \cap S \neq \emptyset\}$ the induced partition on S . A coalition S is connected if $|B(S)| = 1$ (and therefore $B(S) = S$) and not connected if $|B(S)| > 1$. Given a player $i \in N$ we note B^i the set $B \in \mathcal{B}$ such that $i \in B$. Given a connected coalition S , we note B^S the set $B \in \mathcal{B}$ such that $S \subseteq B$.

A game with coalition structure is a triple (N, v, \mathcal{B}) with (N, v) a cooperative game and (N, \mathcal{B}) a coalition structure. Given a game with coalition structure (N, v, \mathcal{B}) we call block game the cooperative game $(N, v/\mathcal{B})$ with

$$(v/\mathcal{B})(S) = \sum_{B \in B(S)} v(B), \quad \forall S \subseteq N.$$

A sharing rule on Γ is a function ψ which associates with each game $(N, v) \in \Gamma$ a vector $\psi(N, v) \in \mathbb{R}^N$. The Shapley value (Shapley [13]) is noted Sh and defined as

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{i\})], \quad \forall i \in N.$$

A sharing rule for games with coalition structure is a function γ which associates with each game with coalition structure (N, v, \mathcal{B}) a vector $\gamma(N, v, \mathcal{B}) \in$

\mathbb{R}^N . The Aumann-Drèze value (Aumann and Drèze [1]) is an extension of the Shapley value to games with coalition structure. We will note it AD and it is defined originally as

$$AD_i(N, v, \mathcal{B}) = Sh_i(B^i, v|_{B^i}), \forall i \in N.$$

The Aumann-Drèze value can also be defined using the block game (Hamiache [4]) as such

$$AD_i(N, v, \mathcal{B}) = Sh_i(N, v/\mathcal{B}), \forall i \in N.$$

3 Associated game and matrix form

Given a game with coalition structure (N, v, \mathcal{B}) and a positive real parameter τ , we define the associated game $(N, v_\tau^*, \mathcal{B})$ with

$$v_\tau^*(S) = \begin{cases} v(S) + \tau \sum_{j \in B^S \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})] & \text{if } |B(S)| = 1, \\ \sum_{R \in B(S)} v_\tau^*(R) & \text{if } |B(S)| > 1. \end{cases} \quad (1)$$

Note that $v_\tau^*(N) = (v/\mathcal{B})(N)$. If $|\mathcal{B}| = 1$ (i.e. if N is connected), by definition of a coalition structure we have $|B(S)| = 1$ for all $S \subseteq N$ and we obtain

$$v_\tau^*(S) = v(S) + \tau \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})],$$

which coincides with the associated game from Hamiache [7].

For a given game, its associated game represents how the players perceive their situation. A given coalition may want to claim a part of the surplus that can be generated by cooperating with each of the surrounding players. Formally, a coalition S may consider having some claim on at least a part τ of the surpluses $[v(S \cup \{j\}) - v(S) - v(\{j\})]$, generated when cooperating with every other players from its block, $j \in B^S \setminus S$. This behaviour can be considered as a “divide and rule” approach from coalition S and a view of its environment conditioned by the coalition structure.

In this paper we will consider large square matrices of order $2^n - 1$. In order to maintain a cohesive notations and ease the manipulation of these different matrices and vectors we will order the set of coalitions of N . This order will depend on the coalition structure (N, \mathcal{B}) . First we will define a lexicographic order for same size coalitions. Let us consider two coalitions of size ϵ , $K = \{k_1, k_2, k_3, \dots, k_\epsilon\}$ and $L = \{l_1, l_2, l_3, \dots, l_\epsilon\}$ with $k_1 < k_2 < k_3 < \dots < k_\epsilon$ and $l_1 < l_2 < l_3 < \dots < l_\epsilon$. The lexicographic order \prec_{lex} for the set of coalitions of

size ϵ is defined as follows, $K \prec_{lex} L$ if and only if $[k_1 < l_1]$ or [there is a natural number γ , with $1 < \gamma \leq \epsilon$, verifying $k_\eta = l_\eta$ for all $1 \leq \eta < \gamma$, and $k_\gamma < l_\gamma$].

We also introduce an order in \mathcal{B} for the different blocks. Given two blocks $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ we shall say that B_1 precedes B_2 if $\min(B_1) < \min(B_2)$.¹ Now, let us consider two coalitions S and T . We will say that coalition S precedes coalitions T under the coalition structure (N, \mathcal{B}) , denoted $S \prec_{\mathcal{B}} T$ if

- $[|B(S)| = 1 \text{ and } |B(T)| > 1]$,
- or if $[|B(S)| > 1, |B(T)| > 1, \text{ and } s < t]$,
- or if $[|B(S)| > 1, |B(T)| > 1, s = t \text{ and } S \prec_{lex} T]$,
- or if $[|B(S)| = |B(T)| = 1 \text{ and } B^S \prec B^T]$,
- or if $[|B(S)| = |B(T)| = 1, B^S = B^T \text{ and } S \prec_{lex} T]$.

This order with respect to \mathcal{B} allows us to regroup all coalitions of a given block together and relegate the non-connected coalitions to the bottom of the list. For example, given a set $N = \{1, 2, 3, 4\}$ and a partition $\mathcal{B} = \{\{1, 2\}, \{3, 4\}\}$, we obtain the following order for the coalitions :

$\{\{1\}, \{2\}, \{1, 2\}, \{3\}, \{4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$.

In Hamiache [5], the matrix form of the associated game of Hamiache [7] has been introduced. This form relies on the square matrix \mathcal{M}_N of order $2^n - 1$ the rows and columns of which are labelled with respect to the lexicographic order \prec_{lex} .² The elements of \mathcal{M}_N for $\emptyset \neq S \subseteq N$ and $\emptyset \neq T \subseteq N$ are given by

$$\mathcal{M}_N[S, T] = \begin{cases} 1 - |N \setminus S|\tau & \text{if } S = T, \\ \tau & \text{if } s + 1 = t \text{ and } S \subseteq T, \\ -\tau & \text{if } t = 1 \text{ and } T \not\subseteq S, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

This matrix depends on the set N . We know that matrix \mathcal{M}_N is diagonalizable, and that 1 is an eigenvalue with algebraic multiplicity n . We also know that $1 - s\tau$ are eigenvalues of \mathcal{M}_N with multiplicity $\binom{n}{s}$ for all s verifying $2 \leq s \leq n$.

Given a coalition structure (N, \mathcal{B}) we define $Q_{\mathcal{B}}$ a square matrix of order $2^n - 1$ whose elements are defined, for all non-empty coalition $S \subseteq N$, $T \subseteq N$ as such

$$Q_{\mathcal{B}}[S, T] = \begin{cases} 1 & \text{if } T \in B(S), \\ 0 & \text{if } T \notin B(S). \end{cases} \quad (3)$$

¹ We define $\min(B_\alpha)$ as the smallest element of B_α . Formally, given a player $i \in B_\alpha$, $i = \min(B_\alpha)$ if and only if $\forall j \neq i, j \in B_\alpha$ we have $i < j$.

² Note that the two orderings \prec_{lex} and $S \prec_{\mathcal{B}} T$ coincide when $\mathcal{B} = \{N\}$.

where M_{B_α} is the square matrix defined in (2) of order $2^{|B_\alpha|} - 1$ corresponding to the block $B_\alpha \in \mathcal{B}$ for any $\alpha \in \{1, 2, \dots, m\}$.

Proof : Since $M_{\mathcal{B}} = Q_{\mathcal{B}} M_N Q_{\mathcal{B}}$, given a coalition $T \subseteq N$, it is easy to see that if $|B(T)| > 1$ then $M_{\mathcal{B}}[S, T] = 0$ for all $S \subseteq N$.

Let us consider the elements $M_{\mathcal{B}}[S, T]$ for $S \subseteq N$ and $T \subseteq N$ such that $|B(S)| = 1$, $|B(T)| = 1$ and $B^S \neq B^T$. Necessarily, $S \neq T$. Since $|B(S)| = 1$, direct computations show that

$$\begin{aligned} M_{\mathcal{B}}[S, T] &= \sum_{L \subseteq N} \sum_{R \subseteq N} Q_{\mathcal{B}}[S, L] \times M_N[L, R] \times Q_{\mathcal{B}}[R, T] \\ &= \sum_{\substack{R \subseteq N \\ T \in \overline{B}(R)}} M_N[S, R]. \end{aligned}$$

Since $T \in B(R)$ and $T \neq S$ we can conclude that $R \neq S$. If $|T| > 1$ then $\forall R \subseteq N$ we have necessarily $|R| > 1$ and $|R| > |S| + 1$, hence $M_{\mathcal{B}}[S, T] = 0$ by definition of M_N .

If $|T| = 1$ we obtain

$$M_{\mathcal{B}}[S, T] = M_N[S, T] + \sum_{\substack{R \subseteq N \\ R \neq T \\ T \in \overline{B}(R)}} M_N[S, R].$$

Since $R \neq T$ then $|R| > 1$, and since $R \neq S$ the only cases where $M_N[S, R]$ is non-zero is when $|R| = |S| + 1$ and $S \subseteq R$. Since $|T| = 1$, the only non-zero case is when $R = S \cup T$. Therefore, by definition of M_N ,

$$M_{\mathcal{B}}[S, T] = -\tau + \tau = 0.$$

Let us consider the elements $M_{\mathcal{B}}[S, T]$ for $S \subseteq N$ and $T \subseteq N$ such that $|B(S)| = 1$, $|B(T)| = 1$ and $B^S = B^T$. We have

$$M_{\mathcal{B}}[S, T] = \sum_{\substack{R \subseteq N \\ T \in \overline{B}(R)}} M_N[S, R].$$

Let us consider that $S \neq T$. We know that $R \neq S$ and $S \not\subseteq R$ since $T \in B(R)$ and $S \neq T$. In this case, if $R \neq T$ we have $|R| > 1$. By definition of M_N we have $M_{\mathcal{B}}[S, T] = 0 = M_{B^S}[S, T]$. If $R = T$ we have $M_{\mathcal{B}}[S, T] = M_N[S, T] = M_{B^S}[S, T]$.

Now let us consider that $S = T$. We obtain

$$\begin{aligned} M_{\mathcal{B}}[S, S] &= \sum_{\substack{R \subseteq N \\ S \in \overline{B}(R)}} M_N[S, R] = \sum_{\substack{R \subseteq N \\ S \in \overline{B}(R) \\ S \neq R}} M_N[S, R] + M_N[S, S] \\ &= \sum_{\substack{R \subseteq N \\ S \in \overline{B}(R) \\ S \neq R \\ |R|=|S|+1}} M_N[S, R] + M_N[S, S] = (n - |B^S|)\tau + 1 - (n - s)\tau \\ &= 1 - (|B^S| - s)\tau = M_{B^S}[S, S], \end{aligned}$$

where the third, fourth and fifth equalities are obtained from the definition of \mathcal{M}_N . \square

Corollary : *The eigenvalues of matrix $M_{\mathcal{B}}$ are 0 with algebraic multiplicities κ , 1 with algebraic multiplicity n , and $1 - s\tau$ with algebraic multiplicity $\binom{|B_\alpha|}{s}$ for all s such that $2 \leq s \leq |B_\alpha|$, for all α in $\{1, 2, \dots, m\}$.*

Proof : It is well known that the eigenvalues of a block triangular matrix are the eigenvalues of diagonal blocks. Knowing the eigenvalues of \mathcal{M}_{B_α} for any $\alpha \in [1, m]$ we obtain the above result. \square

Even though they are irrelevant in the study of the power of matrix $M_{\mathcal{B}}$ and its convergence, we make the following remarks for the sake of completeness. First we illustrate the eigenvectors associated with the eigenvalues 1 of matrix $M_{\mathcal{B}}$. Second, we explicit the elements of the submatrix $\mathbf{C}_{\kappa \times \ell}$.

Remark 1 : For all $i \in N$, the pairs $\langle 1, x_{\{i\}} \rangle$ are eigenpairs of matrix $M_{\mathcal{B}} = Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}}$.

Proof : The proof is identical to the proof of Lemma 6 found in Hamiache & Navarro ([3]). \square

Remark 2 : Submatrix $\mathbf{C}_{\kappa \times \ell}$ is such that

$$C[S, T] = \begin{cases} 1 - |B^T \setminus T| \tau & \text{if } T \in B(S), \\ -\tau & \text{if } |T| = 1, S \cap T = \emptyset, \text{ and } \exists L \in B(S) \\ & \text{such that } B^L = B^T, \\ \tau & \text{if } S \cap T \neq \emptyset \text{ and } \exists L \in B(S) \\ & \text{such that } L \subsetneq T \text{ with } |T| = |L| + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

for all $S, T \subseteq N$ such that $|B(S)| > 1$ and $|B(T)| = 1$.

Proof : In appendix. \square

Lemma 3

For all games with coalition structure (N, v, \mathcal{B}) , a matrix form of the associated game $(N, v_\tau^, \mathcal{B})$ is given by,*

$$v_\tau^* = Q_{\mathcal{B}} \times \mathcal{M}_N \times Q_{\mathcal{B}} \times v.$$

Proof :

$$\begin{aligned}
(Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}} v)[S] &= (Q_{\mathcal{B}} \mathcal{M}_N v/\mathcal{B})[S] \\
&= \sum_{\substack{L \subseteq N \\ L \subseteq N}} \sum_{\substack{R \subseteq N \\ R \subseteq N}} Q_{\mathcal{B}}[S, R] \mathcal{M}_N[R, L] (v/\mathcal{B})(L) \\
&= \sum_{\substack{R \subseteq N \\ R \subseteq N}} Q_{\mathcal{B}}[S, R] \sum_{\substack{L \subseteq N \\ L \subseteq N}} \mathcal{M}_N[R, L] (v/\mathcal{B})(L).
\end{aligned}$$

Using the fact that $(\mathcal{M}_N v)[R] = v(R) + \tau \sum_{j \notin R} [v(R \cup \{j\}) - v(R) - v(\{j\})]$ for any v (Hamiache [7]),

$$\begin{aligned}
&= \sum_{\substack{R \subseteq N \\ R \subseteq N}} Q_{\mathcal{B}}[S, R] \left[(v/\mathcal{B})(R) + \tau \sum_{j \in N \setminus R} [(v/\mathcal{B})(R \cup \{j\}) - (v/\mathcal{B})(R) - (v/\mathcal{B})(\{j\})] \right] \\
&= \sum_{\substack{R \subseteq N \\ R \subseteq N}} (v/\mathcal{B})(R) + \tau \sum_{j \in N \setminus R} [(v/\mathcal{B})(R \cup \{j\}) - (v/\mathcal{B})(R) - (v/\mathcal{B})(\{j\})].
\end{aligned}$$

Since $R \in \mathcal{B}(S)$, R is necessarily connected hence for all $j \notin B^R$ we have

$$(v/\mathcal{B})(R \cup \{j\}) - (v/\mathcal{B})(R) - (v/\mathcal{B})(\{j\}) = 0.$$

Therefore we obtain,

$$\begin{aligned}
(Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}} v)[S] &= \sum_{\substack{R \subseteq N \\ R \subseteq N}} v(R) + \tau \sum_{j \in B^R \setminus R} [v(R \cup \{j\}) - v(R) - v(\{j\})] \\
&= \sum_{\substack{R \subseteq N \\ R \subseteq N}} v_{\tau}^*(R),
\end{aligned}$$

which concludes this proof. \square

Following lemma 3, we can express the sequence of successive associated games as

$$\begin{aligned}
v_{\tau}^* &= (Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}})v, \\
(v_{\tau}^*)_{\tau}^* &= (Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}})v_{\tau}^* = (Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}})^2 v, \\
&\vdots \\
v_{\tau}^{(k^*)} &= (Q_{\mathcal{B}} \mathcal{M}_N Q_{\mathcal{B}})^k v \\
v_{\tau}^{(k^*)} &= (M_{\mathcal{B}})^k v.
\end{aligned}$$

Lemma 4

The sequence of matrices $\{(M_{\mathcal{B}})^k\}_{k=1}^{\infty}$ is convergent for $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_{\alpha}|}$.

We note $\widetilde{M}_{\mathcal{B}}$ its limit.

Proof : From lemma 2 we know

$$M_{\mathcal{B}} = \left[\begin{array}{c|c} \mathbf{A}_{\ell \times \ell} & \mathbf{0}_{\ell \times \kappa} \\ \hline \mathbf{C}_{\kappa \times \ell} & \mathbf{0}_{\kappa \times \kappa} \end{array} \right].$$

with A the matrix such that

$$\mathbf{A}_{\ell \times \ell} = \left[\begin{array}{ccc} \mathcal{M}_{B_1} & & \\ & \mathcal{M}_{B_2} & 0 \\ & & \ddots \\ & 0 & & \ddots \\ & & & & \mathcal{M}_{B_m} \end{array} \right].$$

Direct computations show that

$$(M_{\mathcal{B}})^k = \left[\begin{array}{c|c} \mathbf{A}^k & \mathbf{0}_{\ell \times \kappa} \\ \hline \mathbf{C} \times \mathbf{A}^{k-1} & \mathbf{0}_{\kappa \times \kappa} \end{array} \right].$$

with

$$\mathbf{A}^k = \left[\begin{array}{ccc} (\mathcal{M}_{B_1})^k & & \\ & (\mathcal{M}_{B_2})^k & 0 \\ & & \ddots \\ & 0 & & \ddots \\ & & & & (\mathcal{M}_{B_m})^k \end{array} \right].$$

From Hamiache ([5]) we know that each sequence of matrices $\{(\mathcal{M}_{B_{\alpha}})^k\}_{k=1}^{\infty}$ is convergent if $0 < \tau < \frac{2}{|B_{\alpha}|}$, hence the sequence of matrices $\{A^k\}_{k=1}^{\infty}$ is also convergent, for $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_{\alpha}|}$. We can conclude that the sequence of matrices $\{(M_{\mathcal{B}})^k\}_{k=1}^{\infty}$ is also convergent for $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_{\alpha}|}$.³

□

³ Here the reader will observe that, except for the case where $\mathcal{B} = \{N\}$, τ belongs to a larger interval than in Hamiache ([7]).

Corollary 1: *The sequence of associated games $\{v_\tau^{(k^*)}\}_{k=1}^\infty$ is convergent for $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_\alpha|}$. We note \tilde{v} its limit.*

Corollary 2: The matrix $\widetilde{M_B}$ is of the form

$$(M_B)^k = \left[\begin{array}{c|c} \mathbf{A}^k & \mathbf{0}_{\ell \times \kappa} \\ \hline \mathbf{C} \times \mathbf{A}^{k-1} & \mathbf{0}_{\kappa \times \kappa} \end{array} \right].$$

with

$$\mathbf{A}^k = \left[\begin{array}{cccc} (\mathcal{M}_{B_1})^k & & & \\ & (\mathcal{M}_{B_2})^k & 0 & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \\ & & & (\mathcal{M}_{B_m})^k \end{array} \right].$$

Lemma 5

The limit game \tilde{v} of the sequence of associated games is inessential.

Proof : For any block $B_\alpha \in \mathcal{B}$, we note $\widetilde{\mathcal{M}_{B_\alpha}}$ the limit of the sequence of matrices $\{(\mathcal{M}_{B_\alpha})^k\}_{k=1}^\infty$. For any coalitions $S \subseteq B_\alpha$, using lemma 4 we obtain

$$\begin{aligned} \tilde{v}(S) &= (\widetilde{\mathcal{M}_B} v)[S] = \sum_{\substack{T \\ T \subseteq N}} \widetilde{\mathcal{M}_B}[S, T] v(T) \\ &= \sum_{\substack{T \\ T \subseteq B_\alpha}} \widetilde{\mathcal{M}_B}[S, T] v(T) + \sum_{\substack{T \\ T \not\subseteq B_\alpha}} \widetilde{\mathcal{M}_B}[S, T] v(T). \end{aligned}$$

Since $S \subseteq B_\alpha$, using corollary 2 of lemma 4 we can conclude that $\widetilde{\mathcal{M}_B}[S, T] = 0$ when $T \not\subseteq B_\alpha$ and that $\widetilde{\mathcal{M}_B}[S, T] = \widetilde{\mathcal{M}_{B_\alpha}}[S, T]$ when $T \subseteq B_\alpha$. Hence,

$$\tilde{v}(S) = \sum_{\substack{T \\ T \subseteq B_\alpha}} \widetilde{\mathcal{M}_{B_\alpha}}[S, T] v(T).$$

From Hamiache ([5], Result 4) we know that the limit game is inessential when communication is complete. Since $B_\alpha \in \mathcal{B}$ communication is complete inside B_α , thus

$$= \sum_{\substack{T \\ T \subseteq B_\alpha}} \widetilde{\mathcal{M}_{B_\alpha}}[S, T] v(T) = \sum_{j \in S} \tilde{v}(\{j\}).$$

Let us now consider S such that $|B(S)| > 1$. Since $(M_{\mathcal{B}})^k = (Q_B \mathcal{M}_N Q_B)^{k-1} \times (Q_B \mathcal{M}_N Q_B)$, by idempotence of Q_B we know that $(M_{\mathcal{B}})^k v = (M_{\mathcal{B}})^k (v/\mathcal{B})$. Hence, by definition of v/\mathcal{B} we have

$$\tilde{v}(S) = (\tilde{v}/\mathcal{B})(S) = \sum_{T \in \mathcal{B}(S)} \tilde{v}(T) = \sum_{T \in \mathcal{B}(S)} \sum_{j \in T} \tilde{v}(\{j\}) = \sum_{j \in S} \tilde{v}(\{j\}),$$

which concludes our proof. \square

4 Axioms and characterization

We consider the following system of axioms :

Axiom 1 (*Inessential Game*) :

For all inessential games (N, v) , the sharing rule γ verifies $\gamma_i(N, v, \mathcal{B}) = v(\{i\})$ for all $i \in N$.

Axiom 2 (*Associated Consistency*) :

For all games with coalition structure (N, v, \mathcal{B}) and its associated game $(N, v_{\tau}^*, \mathcal{B})$, the sharing rule γ verifies $\gamma(N, v, \mathcal{B}) = \gamma(N, v_{\tau}^*, \mathcal{B})$.

Axiom 3 (*Continuity*) :

For all convergent sequences $\{(N, v_k, \mathcal{B})\}_{k=1}^{\infty}$ the limit of which is game $(N, \tilde{v}, \mathcal{B})$ we have $\lim_{k \rightarrow \infty} \gamma(N, v_k, \mathcal{B}) = \gamma(N, \tilde{v}, \mathcal{B})$. (The convergence of the games is pointwise).

This system of axiom is a direct adaptation to games with coalition structure of the set of axioms used in Hamiache [7] to characterize the Shapley value. This adaptation is similar to the one done in Hamiache and Navarro [3] for games with graphs.

We can now enounce the main result of this paper :

Theorem 1

There is only one sharing rule γ for games with communication structure that satisfies axioms 1 to 3, provided that $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_{\alpha}|}$ and it is such that $\gamma = AD$.

Proof : We will first prove that the Aumann-Drèze value satisfies axioms 1 to 3.

Inessential game axiom

Given an inessential game (N, v, \mathcal{B}) we have

$$AD_i(N, v, \mathcal{B}) = Sh_i(N, v/\mathcal{B}) = (v/\mathcal{B})(\{i\}) = v(\{i\}),$$

where the third equality comes from the Shapley value satisfying the inessential game axiom as defined in Hamiache ([7]). We proved that AD satisfies the inessential game axiom.

Associated consistency axiom

We have

$$AD_i(N, v_\tau^*, \mathcal{B}) = Sh_i(B^i, (v_\tau^*)_{|B^i}),$$

from the definition of AD . Let us consider a coalition $S \subseteq B^i$. We first remark that if $S \subseteq B^i$ then $B^S = B^i$. On one hand we have

$$(v_\tau^*)_{|B^i}(S) = (v_\tau^*)(S)$$

by definition of a sub-game restricted to coalition B^i . On the other hand,

$$\begin{aligned} (v_{|B^i})_\tau^*(S) &= v_{|B^i}(S) + \tau \sum_{j \in B^S \setminus S} [v_{|B^i}(S \cup \{j\}) - v_{|B^i}(S) - v_{|B^i}(\{j\})] \\ &= v(S) + \tau \sum_{j \in B^S \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})] \\ &= (v_\tau^*)(S). \end{aligned}$$

The first and last equalities come from the definition of the associated game and the second from the definition of a sub-game restricted to coalition B^i . Hence

$$(v_\tau^*)_{|B^i}(S) = (v_{|B^i})_\tau^*(S), \quad \forall S \subseteq B^i.$$

Consequently we have

$$Sh_i(B^i, (v_\tau^*)_{|B^i}) = Sh_i(B^i, (v_{|B^i})_\tau^*).$$

Since $B^S = B^i$ we can observe that the associated game $(B^i, (v_{|B^i})_\tau^*, \mathcal{B})$ would coincides with an associated game $(B^i, (v_{|B^i})_\tau^*)$ as defined in Hamiache ([7]). Using the fact that the Shapley value satisfies the associated consistency axiom defined in the aforementioned paper we have

$$\begin{aligned} Sh_i(B^i, (v_{|B^i})_\tau^*) &= Sh_i(B^i, v_{|B^i}) \\ &= AD_i(N, v, \mathcal{B}), \end{aligned}$$

where the last equality comes from the definition of AD . We proved that AD satisfies the associated consistency axiom.

Continuity

For any convergent sequences $\{(N, v_k, \mathcal{B})\}_{k=1}^{\infty}$ the limit of which is game $(N, \tilde{v}, \mathcal{B})$ we have that

$$\lim_{k \rightarrow \infty} AD_i(N, v_k, \mathcal{B}) = \lim_{k \rightarrow \infty} Sh_i(B^i, (v_k)_{|_{B^i}}) = Sh_i(B^i, \tilde{v}_{|_{B^i}}) = AD_i(N, \tilde{v}, \mathcal{B}),$$

which proves that AD satisfies continuity.

Unicity

Now for the unicity part, by associated consistency, continuity, lemma 5 and the inessential game axioms we have

$$\gamma_i(N, v, \mathcal{B}) = \lim_{k \rightarrow \infty} \gamma_i(N, v_{\tau}^{(k*)}, \mathcal{B}) = \gamma_i(N, \tilde{v}, \mathcal{B}) = \tilde{v}(\{i\}),$$

for any $i \in N$.

Using lemma 5 and the fact that the Shapley value satisfies the inessential game axiom we have that

$$\tilde{v}(\{i\}) = Sh_i(B^i, \tilde{v}_{|_{B^i}}).$$

Since communication is complete inside a block, we have that the associated game defined in (3) coincides with the associated game from Hamiache ([7]). Using the fact that the Shapley value satisfies the associated game axiom from ([7]) we have that, for $0 < \tau < \frac{2}{\max_{\alpha \in [1, m]} |B_{\alpha}|}$,

$$Sh_i(B^i, \tilde{v}_{|_{B^i}}) = Sh_i(B^i, v_{|_{B^i}}).$$

We can therefore conclude that, for any $i \in N$ we have

$$\gamma_i(N, v, \mathcal{B}) = \tilde{v}(\{i\}) = Sh_i(B^i, \tilde{v}_{|_{B^i}}) = Sh_i(B^i, v_{|_{B^i}}) = AD_i(N, v, \mathcal{B}),$$

the last equality coming from the definition of AD . □

5 Conclusion

The present work joins Hamiache [7] and Hamiache and Navarro [3] in characterizing a sharing rule with the same set of axioms. If associated consistency, continuity and inessential game axioms are considered desirable we know that the Shapley value for games with perfect communication, the Hamiache-Navarro value for games with graphs and the Aumann-Drèze value for games with coalition structures are the three incarnation of this set of axioms. The

development of such characterizations spanning different types of problems appear to be an interesting angle for future research as it allows to define a desirable set of axioms to apply regardless of the context in which cooperation occurs and be assured that a unique solution satisfies this set of axioms.

In addition this article provides a missing link between the Myerson value, the Shapley value and the Hamiache-Navarro value. On one hand, it is well-known that the Aumann-Drèze value can be expressed as the Myerson value of a game with a graph whose complete components are the blocks of the coalition structures. On the other hand, the present article offers a characterization of the Aumann-Drèze value that relies on the same axioms used to characterize the Shapley value and the Hamiache-Navarro value. Considering that the results given by the Myerson value and the Hamiache-Navarro value differ, we now know that these two sharing rules are two different extensions of the Aumann-Drèze value as well as the Shapley value. The Hamiache-Navarro value is therefore as legitimate with respect to the existing literature as is the Myerson value which strengthens the results presented in Hamiache and Navarro [3].

Appendix

Proof of Remark 2: Let us consider all coalitions $S, T \subseteq N$ with $|B(S)| > 1$ and $|B(T)| = 1$. We have

$$\begin{aligned} C[S, T] &= M_B[S, T] = (Q_B \mathcal{M}_N Q_B)[S, T] \\ &= \sum_{\substack{L \\ L \subseteq N}} \sum_{\substack{R \\ R \subseteq N}} Q_B[S, L] \mathcal{M}_N[L, R] Q_B[R, T] \\ &= \sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \\ R \subseteq N}} \mathcal{M}_N[L, R] Q_B[R, T] \\ &= \sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \\ T \in B(R)}} \mathcal{M}_N[L, R] \end{aligned}$$

From here on we will split the proof into two different cases: $T \in B(S)$ and $T \notin B(S)$.

Case 1:

We assume $T \in B(S)$. Let us decompose the sums.

$$\begin{aligned} C[S, T] &= \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[T, R] + \sum_{\substack{L \neq T \\ L \in B(S)}} \sum_{\substack{R \\ T \in B(R)}} \mathcal{M}_N[L, R] + \mathcal{M}_N[T, T] \\ &= \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[T, R] + \sum_{\substack{L \neq T \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] \\ &\quad + \sum_{\substack{L \neq T \\ L \in B(S)}} \mathcal{M}_N[L, T] + \mathcal{M}_N[T, T]. \end{aligned}$$

We will study each term of this expression. Let us consider the first term. Every $R \neq T$ such that $T \in B(R)$ are necessarily unions of T with coalitions of players $\notin B^T$. Consequently,

$|R| > 1$. Hence, by definition of \mathcal{M}_N , the term $\mathcal{M}_N[T, R]$ is non-zero if and only if $|R| = |T| + 1$. We obtain

$$\sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[T, R] = |N \setminus B^T| \tau.$$

We consider now the second term. If $L \in B(S)$ and $L \neq T$ then $L \cap T = \emptyset$ since $T \in B(S)$. As in the first term we also know that coalitions R are unions of T with coalitions of players $\notin B^T$ and that $|R| > 1$. The term $\mathcal{M}_N[L, R]$ is therefore non-zero if and only when $|R| = |L| + 1$ and $L \subseteq R$.

However, since R is necessarily the union of T and coalition $\notin B^T$ the only cases where $|R| = |L| + 1$ and $L \subseteq R$ is when $|T| = 1$ and $R = L \cup T$. If $|T| = 1$, for each $L \in B(S)$, $L \neq T$ we have one and only one $R = L \cup T$. We obtain

$$\sum_{\substack{L \neq T \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] = \begin{cases} (|B(S)| - 1) \tau & \text{if } |T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We consider now the third term. If $L \in B(S)$ and $L \neq T$ then $L \cap T = \emptyset$. Hence, the only case when $\mathcal{M}_N[L, T]$ is non-zero is if $|T| = 1$ since $T \not\subseteq L$. We obtain

$$\sum_{\substack{L \neq T \\ L \in B(S)}} \mathcal{M}_N[L, T] = \begin{cases} -\tau (|B(S)| - 1) & \text{if } |T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the second and third term cancel each other. We obtain

$$\begin{aligned} C[S, T] &= |N \setminus B^T| \tau + \mathcal{M}_N[T, T] \\ &= |N \setminus B^T| \tau + 1 - |N \setminus T| \tau \\ &= 1 - |B^T \setminus T| \tau. \end{aligned}$$

Which concludes the first case.

Case 2:

We assume $T \notin B(S)$. Let us decompose the sums.

$$C[S, T] = \sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] + \sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T].$$

Since $R \neq T$ and $T \in B(R)$ we have $|B(R)| > 1$ and thus $R \neq L$ in the first term. Since $L \in B(S)$ and $T \notin B(S)$ we have $L \neq T$ in the second term. Consequently we eliminate the possibility of the first non-zero case in expression (2).

We can split our current case into two subcases: $S \cap T = \emptyset$ and $S \cap T \neq \emptyset$.

Subcase 2.1:

We assume $S \cap T = \emptyset$. Hence, we have that $L \not\subseteq T$ when $L \in B(S)$. We also have that $|R| > 1$ if $R \neq T$ and $T \in B(R)$. Therefore, in this case, when $L \in B(S)$ and $R \neq T$, $T \in B(R)$, both $\mathcal{M}_N[L, R]$ and $\mathcal{M}_N[L, T]$ are potentially non-zero only if $|T| = 1$.

Let us consider that $\exists L \in B(S)$ such that $B^L = B^T$ (meaning that T belongs to the same block as some component of S). By definition there can only be one such component of S . Consequently $\exists R$, $R \neq T$, $T \in B(R)$ such that $R = L \cup T$, $|R| = |L| + 1$ for every

$L \in B(S)$ except the one that belongs to the same block as T . Hence

$$\begin{aligned} C[S, T] &= \sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] + \sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T] \\ &= (|B(S)| - 1)\tau + \sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T]. \end{aligned}$$

Since $|T| = 1$ and $L \not\subseteq T$

$$\begin{aligned} C[S, T] &= (|B(S)| - 1)\tau + \sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T] \\ &= (|B(S)| - 1)\tau + |B(S)|\tau = -\tau. \end{aligned}$$

If however $B^L \neq B^T$ for all $L \in B(S)$ therefore $\exists R, R \neq T, T \in B(R)$ such that $R = L \cup T, |R| = |L| + 1$ for every $L \in B(S)$. We obtain

$$\begin{aligned} C[S, T] &= \sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] + \sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T] \\ &= |B(S)|\tau + |B(S)|\tau = 0, \end{aligned}$$

which concludes subcase 2.1.

Subcase 2.2:

We assume $S \cap T \neq \emptyset$. We will also split this subcase into two further subcases. Knowing that $|B(T)| = 1$, we can consider only two situations when $S \cap T \neq \emptyset$. Either T belong entirely to S in which case it belongs entirely to a component of S . In other words $\exists L \in B(S)$ such that $T \subseteq L$. The second case is when T overlaps with one component of S , in other words $T \not\subseteq S$ with $S \cap T \neq \emptyset$.

Let us consider first that $\exists L \in B(S)$ such that $T \subseteq L$. This implies that $T \subseteq S$. We have $|R| > 1$ for $R \neq T, T \in B(S)$ hence $\mathcal{M}_N[L, R]$ is non-zero for $L \in B(S)$ if and only if $|R| = |L| + 1$ and $L \subseteq R$. This is the case only if $R = L \cup T$ with $|T| = 1$. Since $\exists L \in B(S)$ such that $T \subseteq L$ then $R = L \cup T$ is connected for this particular L and as such $T \notin B(R)$. We obtain

$$\sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] = (|B(S)| - 1)\tau,$$

when $|T| = 1$ and 0 otherwise. For $\sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T]$, since $T \subseteq L$, the only situations where $\mathcal{M}_N[L, T]$ is non-zero is if $|T|=1$. In addition, we have $T \not\subseteq L$ for every other $L \in B(S)$. Hence

$$\sum_{\substack{L \\ L \in B(S)}} \mathcal{M}_N[L, T] = |B(S) - 1|\tau,$$

when $|T| = 1$ and 0 otherwise. We obtain

$$C[S, T] = 0.$$

Let us now consider that $T \not\subseteq S$. Knowing that $S \cap T \neq \emptyset$ we know that $|T| > 1$. If $R \neq T$ and $T \in B(R)$ it is impossible to have $|R| = 1$ or $|R| = |L| + 1$ since $|T| > 1$. Hence

$$\sum_{\substack{L \\ L \in B(S)}} \sum_{\substack{R \neq T \\ T \in B(R)}} \mathcal{M}_N[L, R] = 0.$$

Considering the term $\sum_{L \in B(S)} \mathcal{M}_N[L, T]$, we have $\mathcal{M}_N[L, T] \neq 0$ if and only if $|T| = |L| + 1$ and $L \subseteq T$. This situation arises only if $\exists L \in B(S)$ such that $L \subsetneq T$. We obtain

$$C[S, T] = \tau,$$

if $|T| = |L| + 1$, $L \in B(S)$ and $L \subsetneq T$. Which concludes subcase 2.2.

Going over all the cases we conclude that

$$C[S, T] = \begin{cases} 1 - |B^T \setminus T| \tau & \text{if } T \in B(S), \\ -\tau & \text{if } |T| = 1, S \cap T = \emptyset, \text{ and } \exists L \in B(S) \\ & \text{such that } B^L = B^T, \\ \tau & \text{if } S \cap T \neq \emptyset \text{ and } \exists L \in B(S) \\ & \text{such that } L \subsetneq T \text{ with } |T| = |L| + 1, \\ 0 & \text{otherwise,} \end{cases}$$

□

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