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# An algebraic index theorem for Poisson manifolds

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**Abstract.** The formality theorem for Hochschild chains of the algebra of functions on a smooth manifold gives us a version of the trace density map from the zeroth Hochschild homology of a deformation quantization algebra to the zeroth Poisson homology. We propose a version of the algebraic index theorem for a Poisson manifold which is based on this trace density map.

## 1. Introduction

Various versions [5], [6], [11], [13], [18], [19], [33] of the algebraic index theorem generalize the famous Atiyah-Singer index theorem [1] from the case of a cotangent bundle to an arbitrary symplectic manifold. The first version of this theorem for Poisson manifolds was proposed by D. Tamarkin and B. Tsygan in [43]. Unfortunately, the proof of this version is based on the formality conjecture for cyclic chains [44] which is not yet established. In this paper we use the formality theorem for Hochschild chains [15], [41] to prove another version of the algebraic index theorem for an arbitrary Poisson manifold. This version is based on the trace density map from the zeroth Hochschild homology of the deformation quantization algebra to the zeroth Poisson homology of the manifold.

We denote by  $(M, \pi_1)$  a smooth real Poisson manifold and by  $\mathcal{O}_M$  the algebra of smooth (real-valued) functions on  $M$ .  $TM$  (resp.  $T^*M$ ) stands for tangent (resp. cotangent) bundle of  $M$ .

Let  $\mathcal{O}_M^{\hbar} = (\mathcal{O}_M[[\hbar]], *)$  be a deformation quantization algebra of  $(M, \pi_1)$  in the sense of [2] and [3] and let

$$(1.1) \quad \pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\Gamma(M, \wedge^2 TM)[[\hbar]]$$

be a representative of Kontsevich's class of  $\mathcal{O}_M^{\hbar}$ .

One of the versions of the algebraic index theorem for a symplectic manifold [11] describes a natural map (see [11], Eq. (31) and Theorem 4)

$$cl : K_0(\mathcal{O}_M^{\hbar}) \rightarrow H_{\text{DR}}^{\text{top}}(M)(\hbar)$$

from the  $K$ -theory of the deformation quantization algebra  $\mathcal{O}_M^{\hbar}$  to the top degree De Rham cohomology of  $M$ . This map is obtained by composing the trace density map [19]

$$(1.2) \quad \text{trd}_{\text{symp}} : \mathcal{O}_M^{\hbar}/[\mathcal{O}_M^{\hbar}, \mathcal{O}_M^{\hbar}] \rightarrow H_{\text{DR}}^{\text{top}}(M)((\hbar))$$

from the zeroth Hochschild homology  $HH_0(\mathcal{O}_M^{\hbar}) = \mathcal{O}_M^{\hbar}/[\mathcal{O}_M^{\hbar}, \mathcal{O}_M^{\hbar}]$  of  $\mathcal{O}_M^{\hbar}$  to the top degree De Rham cohomology of  $M$  with the lowest component of the Chern character (see [30], Example 8.3.6)

$$(1.3) \quad \text{ch}_{0,0} : K_0(\mathcal{O}_M^{\hbar}) \rightarrow \mathcal{O}_M^{\hbar}/[\mathcal{O}_M^{\hbar}, \mathcal{O}_M^{\hbar}]$$

from the  $K$ -theory of  $\mathcal{O}_M^{\hbar}$  to the zeroth Hochschild homology of  $\mathcal{O}_M^{\hbar}$ .

In the case of an arbitrary Poisson manifold one cannot construct the map (1.2). Instead, the formality theorem for Hochschild chains [15], [41] provides us with the map

$$(1.4) \quad \text{trd} : \mathcal{O}_M^{\hbar}/[\mathcal{O}_M^{\hbar}, \mathcal{O}_M^{\hbar}] \rightarrow HP_0(M, \pi),$$

from the zeroth Hochschild homology of  $\mathcal{O}_M^{\hbar}$  to the zeroth Poisson homology [7], [26] of  $\pi$  (1.1).

According to J.-L. Brylinski [7], if  $(M, \pi_1)$  is a symplectic manifold then we have the following isomorphism:

$$HP_{\bullet}(M, \pi)[\hbar^{-1}] \cong H_{\text{DR}}^{\dim M - \bullet}(M)((\hbar)).$$

Thus, in the symplectic case the map (1.2) can be obtained from the map (1.4).

For this reason we also refer to (1.4) as *the trace density map*.

Composing (1.4) with (1.3) we get the map

$$(1.5) \quad \text{ind} : K_0(\mathcal{O}_M^{\hbar}) \rightarrow HP_0(M, \pi).$$

Let us call this map *the quantum index density*.

On the other hand setting  $\hbar = 0$  gives us the obvious map

$$(1.6) \quad \sigma : K_0(\mathcal{O}_M^{\hbar}) \rightarrow K_0(\mathcal{O}_M)$$

which we call *the principal symbol map*.

We recall that

**Proposition 1** (J. Rosenberg, [39]). *The map  $\text{ind}$  (1.5) factors through the map  $\sigma$  (1.6).*

The proof of this proposition is nice and transparent. For this reason we decided to recall it here in the introduction.

*Proof.* First, recall that for every associative algebra  $B$  a finitely generated projective module can be represented by an idempotent in the algebra  $\text{Mat}(B)$  of finite size matrices with entries in  $B$ .

Next, let us show that the map (1.6) is surjective. To do this, it suffices to show that for every idempotent  $q$  in  $\text{Mat}(\mathcal{O}_M)$  there exists an idempotent  $Q$  in  $\text{Mat}(\mathcal{O}_M^{\hbar})$  such that

$$Q|_{\hbar=0} = q.$$

The desired idempotent is produced by the following equation (see [18], Eq. (6.1.4), p. 185):

$$(1.7) \quad Q = \frac{1}{2} + \left( q - \frac{1}{2} \right) * \left( 1 + 4(q * q - q) \right)^{-1/2}$$

where the last term in the right-hand side is understood as the expansion of the function  $y = x^{-1/2}$  in  $4(q * q - q)$  around the point  $(x = 1, y = 1)$ . Since  $q$  is an idempotent in  $\text{Mat}(\mathcal{O}_M)$

$$q * q - q = 0 \text{ mod } \hbar$$

and therefore the expansion in (1.7) makes sense.

A direct computation shows that the element  $Q$  defined by (1.7) is indeed an idempotent in  $\text{Mat}(\mathcal{O}_M^{\hbar})$ .

Thus, it suffices to show that if two idempotents  $P$  and  $Q$  in  $\text{Mat}(\mathcal{O}_M^{\hbar})$  have the same principal part then  $\text{ind}([P]) = \text{ind}([Q])$ . Here  $[P]$  (resp.  $[Q]$ ) denotes the class in  $K_0(\mathcal{O}_M^{\hbar})$  represented by  $P$  (resp.  $Q$ ).

For this, we first show that if

$$(1.8) \quad P|_{\hbar=0} = Q|_{\hbar=0}$$

then  $P$  can be connected to  $Q$  by a smooth path  $P_t$  of idempotents in  $\text{Mat}(\mathcal{O}_M^{\hbar})$ .

Indeed if we define the following smooth path

$$P_t^0 = ((1 - t)P + tQ)$$

in the algebra  $\text{Mat}(\mathcal{O}_M^{\hbar})$  and plug  $P_t^0$  into Equation (1.7) instead of  $q$  we get the path of idempotents in the algebra  $\text{Mat}(\mathcal{O}_M^{\hbar})$

$$(1.9) \quad P_t = \frac{1}{2} + \left( P_t^0 - \frac{1}{2} \right) * \left( 1 + 4(P_t^0 * P_t^0 - P_t^0) \right)^{-1/2},$$

which connects  $P$  with  $Q$ . Due to Equation (1.8) the right-hand side of (1.9) is well defined as a formal power series in  $\hbar$ .

Let us now show that  $P$  and  $Q$  represent the same class in  $K_0(\mathcal{O}_M^{\hbar})$ . Since

$$\begin{aligned}d_t P_t &= (d_t P_t) * P_t + P_t * (d_t P_t), \\P_t * (d_t P_t) * P_t &= 0.\end{aligned}$$

Hence, for  $P_t$  we have

$$d_t P(t) = [P(t), P(t) * (d_t P(t)) - (d_t P(t)) * P(t)].$$

Therefore, since  $P_t$  connects  $P$  and  $Q$ ,  $[P] = [Q]$  in  $K_0(\mathcal{O}_M^{\hbar})$  and the proposition follows.  $\square$

In this paper we propose a version of the algebraic index theorem which describes how the quantum index density (1.5) factors through the principal symbol map (1.6).

More precisely, using the generalization [4], [16] of the formality theorem for Hochschild chains [15], [41] to the algebra of endomorphisms of a vector bundle, we construct the map

$$(1.10) \quad \text{ind}_c : K_0(\mathcal{O}_M) \rightarrow HP_0(M, \pi),$$

which makes the following diagram

$$(1.11) \quad \begin{array}{ccc}K_0(\mathcal{O}_M^{\hbar}) & \xrightarrow{\text{ind}} & HP_0(M, \pi) \\ & \searrow \sigma & \nearrow \text{ind}_c \\ & & K_0(\mathcal{O}_M)\end{array}$$

commutative.

In this paper we refer to the map  $\text{ind}_c$  as *the classical index density*.

The organization of the paper is as follows. In the next section we fix notation and recall some results we are going to use in this paper. In section 3 we prove some useful facts about the twisting procedure of DGLAs and DGLA modules by Maurer-Cartan elements. In section 4 we construct trace density map (1.4), quantum (1.5) and classical (1.10) index densities. In section 5 we formulate and prove the main result of this paper (see Theorem 1). The concluding section consists of two parts. In the first part we describe the relation of our result to the Tamarkin-Tsygan version [43] of the algebraic index theorem. In the second part we propose a conjectural version of our index theorem in the context of Rieffel's strict deformation quantization [28] of dual bundle of a Lie algebroid.

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## 2. Preliminaries

In this section we fix notation and recall some results we are going to use in this paper.

For an associative algebra  $B$  we denote by  $\text{Mat}_N(B)$  the algebra of  $N \times N$  matrices over  $B$ . The notation  $C_\bullet(B)$  is reserved for the normalized Hochschild chain complex of  $B$  with coefficients in  $B$

$$(2.1) \quad C_\bullet(B) = C_\bullet(B, B)$$

and the notation  $C^\bullet(B)$  is reserved for the normalized Hochschild cochain complex of  $B$  with coefficients in  $B$  and with shifted grading

$$(2.2) \quad C^\bullet(B) = C^{\bullet+1}(B, B).$$

The Hochschild coboundary operator is denoted by  $\partial$  and the Hochschild boundary operator is denoted by  $b$ . We denote by  $HH^\bullet(B)$  the cohomology of the complex  $(C^\bullet(B), \partial)$  and by  $HH_\bullet(B)$  the homology of the complex  $(C_\bullet(B), b)$ .

It is well known that the Hochschild cochain complex (2.2) carries the structure of a differential graded Lie algebra. The corresponding Lie bracket (see [17], Eq. (3.2), p. 45) was originally introduced by M. Gerstenhaber in [22]. We will denote this bracket by  $[\ , ]_G$ .

The Hochschild chain complex (2.1) carries the structure of a differential graded Lie algebra module over the DGLA  $C^\bullet(B)$ . We will denote the action (see [17], Eq. (3.5), p. 46) of cochains on chains by  $R$ .

The trace map  $\text{tr}$  [30] is the map from the Hochschild chain complex  $C_\bullet(\text{Mat}_N(B))$  of the algebra  $\text{Mat}_N(B)$  to the Hochschild chain complex  $C_\bullet(B)$  of the algebra  $B$ . This map is defined by the formula

$$(2.3) \quad \text{tr}(M_0 \otimes M_1 \otimes \cdots \otimes M_k) = \sum_{i_0, \dots, i_k} (M_0)_{i_0 i_1} \otimes (M_1)_{i_1 i_2} \otimes \cdots \otimes (M_k)_{i_k i_0},$$

where  $M_0, \dots, M_k$  are matrices in  $\text{Mat}_N(B)$  and  $(M_a)_{ij}$  are the corresponding entries.

Dually, the cotrace map [30]

$$\text{cotr} : C^\bullet(B) \rightarrow C^\bullet(\text{Mat}_N(B))$$

is defined by the formula

$$(2.4) \quad (\text{cotr}(P)(M_0, M_1, \dots, M_k))_{ij} = \sum_{i_1, \dots, i_k} P((M_0)_{ii_1}, (M_1)_{i_1 i_2}, \dots, (M_k)_{i_k j}),$$

where  $P \in C^k(B)$  and  $M_0, \dots, M_k$  are, as above, matrices in  $\text{Mat}_N(B)$ .

“DGLA” always means a differential graded Lie algebra. The arrow  $\succ \rightarrow$  denotes an  $L_\infty$ -morphism of DGLAs, the arrow  $\succ \succ \rightarrow$  denotes a morphism of  $L_\infty$ -modules, and the notation

$$\begin{array}{c} \mathcal{L} \\ \downarrow \text{mod} \\ \mathcal{M} \end{array}$$

means that  $\mathcal{M}$  is a DGLA module over the DGLA  $\mathcal{L}$ . The symbol  $\circ$  always stands for the composition of morphisms.  $\hbar$  denotes the formal deformation parameter.

Throughout this paper  $M$  is a smooth real manifold. For a smooth real vector bundle  $E$  over  $M$ , we denote by  $\text{End}(E)$  the algebra of endomorphisms of  $E$ . For a sheaf  $\mathcal{G}$  of  $\mathcal{O}_M$ -modules we denote by  $\Gamma(M, \mathcal{G})$  the vector space of global sections of  $\mathcal{G}$  and by  $\Omega^\bullet(\mathcal{G})$  the graded vector space of exterior forms on  $M$  with values in  $\mathcal{G}$ . In few cases, by abuse of notation, we denote by  $\Omega^\bullet(\mathcal{G})$  the sheaf of exterior forms with values in  $\mathcal{G}$ . Similarly, we sometimes refer to  $\text{End}(E)$  as the sheaf of endomorphisms of a vector bundle  $E$ . We specifically clarify the notation when it is not clear from the context.

$T_{\text{poly}}^\bullet$  is the vector space of polyvector fields with shifted grading

$$T_{\text{poly}}^\bullet = \Gamma(M, \wedge_{\mathcal{O}_M}^{\bullet+1} TM), \quad T_{\text{poly}}^{-1} = \mathcal{O}_M,$$

and  $\mathcal{A}^\bullet$  is the graded vector space of exterior forms:

$$\mathcal{A}^\bullet = \Gamma(M, \wedge_{\mathcal{O}_M}^\bullet T^*M).$$

$T_{\text{poly}}^\bullet$  is the graded Lie algebra with respect the so-called Schouten-Nijenhuis bracket  $[\cdot, \cdot]_{\text{SN}}$  (see [17], Eq. (3.20), p. 50) and  $\mathcal{A}^\bullet$  is the graded Lie algebra module over  $T_{\text{poly}}^\bullet$  with respect to Lie derivative  $L$  (see [17], Eq. (3.21), p. 51). We will regard  $T_{\text{poly}}^\bullet$  (resp.  $\mathcal{A}^\bullet$ ) as the DGLA (resp. the DGLA module) with the zero differential.

Given a Poisson structure  $\pi$  (1.1) one may introduce non-zero differentials on the graded Lie algebra  $T_{\text{poly}}^\bullet[[\hbar]]$  and on the graded Lie algebra module  $\mathcal{A}^\bullet[[\hbar]]$ . Namely,  $T_{\text{poly}}^\bullet[[\hbar]]$  can be equipped with the Lichnerowicz differential  $[\pi, \cdot]_{\text{SN}}$  [29] and  $\mathcal{A}^\bullet[[\hbar]]$  can be equipped with the Koszul differential  $L_\pi$  [26], where  $L$  denotes the Lie derivative. The cohomology of the complex  $(T_{\text{poly}}^\bullet[[\hbar]], [\pi, \cdot]_{\text{SN}})$  is called the Poisson cohomology of  $\pi$ . For these cohomology groups we reserve the notation  $HP^\bullet(M, \pi)$ . Similarly, the homology of the complex  $(\mathcal{A}^\bullet[[\hbar]], L_\pi)$  is called the Poisson homology of  $\pi$  and for the homology groups of  $(\mathcal{A}^\bullet[[\hbar]], L_\pi)$  we reserve notation  $HP_\bullet(M, \pi)$ .

We denote by  $x^i$  local coordinates on  $M$  and by  $y^i$  fiber coordinates in the tangent bundle  $TM$ . Having these coordinates  $y^i$  we can introduce another local basis of exterior forms  $\{dy^i\}$ . We will use both bases  $\{dx^i\}$  and  $\{dy^i\}$ . In particular, the notation  $\Omega^\bullet(\mathcal{G})$  is reserved for the  $dy$ -exterior forms with values in the sheaf  $\mathcal{G}$  while  $\mathcal{A}^\bullet$  is reserved for the  $dx$ -exterior forms.

We now briefly recall the Fedosov resolutions (see [17], Chapter 4) of polyvector fields, exterior forms, and Hochschild (co)chains of  $\mathcal{O}_M$ . This construction has various incarnations and it is referred to as the Gelfand-Fuchs trick [20] or formal geometry [21] in the sense of Gelfand and Kazhdan, or mixed resolutions [45] of Yekutieli.

We denote by  $\mathcal{S}M$  the formally completed symmetric algebra of the cotangent bundle  $T^*(M)$ . Sections of the sheaf  $\mathcal{S}M$  can be viewed as formal power series in tangent coordinates  $y^i$ . We regard  $\mathcal{S}M$  as the sheaf of algebras over  $\mathcal{O}_M$ . In particular,  $C^\bullet(\mathcal{S}M)$  is the sheaf of normalized Hochschild cochains of  $\mathcal{S}M$  over  $\mathcal{O}_M$ . Namely, the sections of  $C^k(\mathcal{S}M)$  over an open subset  $U \subset M$  are  $\mathcal{O}_M$ -linear polydifferential operators with respect to the tangent coordinates  $y^i$

$$P : \Gamma(U, \mathcal{S}M)^{\otimes(k+1)} \rightarrow \Gamma(U, \mathcal{S}M)$$

satisfying the normalization condition

$$P(\dots, f, \dots) = 0, \quad \forall f \in \mathcal{O}_M(U).$$

Similarly,  $C_\bullet(\mathcal{S}M)$  is the sheaf of normalized Hochschild chains<sup>1)</sup> of  $\mathcal{S}M$  over  $\mathcal{O}_M$ . As in [17] the tensor product in

$$C_k(\mathcal{S}M) = \underbrace{\mathcal{S}M \hat{\otimes}_{\mathcal{O}_M} (\mathcal{S}M/\mathcal{O}_M) \hat{\otimes}_{\mathcal{O}_M} \cdots \hat{\otimes}_{\mathcal{O}_M} (\mathcal{S}M/\mathcal{O}_M)}_{k+1}$$

is completed in the adic topology in fiber coordinates  $y^i$  on the tangent bundle  $TM$ .

The cohomology of the complex of sheaves  $C^\bullet(\mathcal{S}M)$  is the sheaf  $\mathcal{F}_{\text{poly}}^\bullet$  of fiberwise polyvector fields (see [17], p. 60). The cohomology of the complex of sheaves  $C_\bullet(\mathcal{S}M)$  is the sheaf  $\mathcal{E}^\bullet$  of fiberwise differential forms (see [17], p. 62). These are  $dx$ -forms with values in  $\mathcal{S}M$ .

In [17], Theorem 4, p. 68, it is shown that the algebra  $\Omega^\bullet(\mathcal{S}M)$  can be equipped with a differential of the following form

$$(2.5) \quad D = \nabla - \delta + A,$$

where

$$(2.6) \quad \nabla = dy^i \frac{\partial}{\partial x^i} - dy^i \Gamma_{ij}^k(x) y^j \frac{\partial}{\partial y^k},$$

<sup>1)</sup> In [17] the sheaf  $C^\bullet(\mathcal{S}M)$  is denoted by  $\mathcal{D}_{\text{poly}}^\bullet$  and the sheaf  $C_\bullet(\mathcal{S}M)$  is denoted by  $\mathcal{E}_{\text{poly}}^\bullet$ .



is a torsion free connection with Christoffel symbols  $\Gamma_{ij}^k(x)$ ,

$$(2.7) \quad \delta = dy^i \frac{\partial}{\partial y^i},$$

and

$$(2.8) \quad A = \sum_{p=2}^{\infty} dy^k A_{ki\dots i}^j(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^j} \in \Omega^1(\mathcal{T}_{\text{poly}}^0).$$

We refer to (2.5) as the Fedosov differential.

Notice that  $\delta$  in (2.7) is also a differential on  $\Omega^\bullet(\mathcal{S}M)$  and (2.5) can be viewed as a deformation of  $\delta$  via the connection  $\nabla$ .

Let us recall from [17] the following operator on<sup>2)</sup>  $\Omega^\bullet(\mathcal{S}M)$ :

$$(2.9) \quad \delta^{-1}(a) = \begin{cases} y^k \frac{\bar{\partial}}{\partial(dy^k)} \int_0^1 a(x, ty, t dy) \frac{dt}{t}, & \text{if } a \in \Omega^{>0}(\mathcal{S}M), \\ 0, & \text{otherwise.} \end{cases}$$

This operator satisfies the following properties:

$$(2.10) \quad \delta^{-1} \circ \delta^{-1} = 0,$$

$$(2.11) \quad a = \chi(a) + \delta \delta^{-1} a + \delta^{-1} \delta a, \quad \forall a \in \Omega^\bullet(\mathcal{S}M)$$

where

$$(2.12) \quad \chi(a) = a|_{y^i = dy^i = 0}.$$

These properties are used in the proof of the acyclicity of  $\delta$  and  $D$  in positive dimension.

According to [17], Proposition 10, p. 64, the sheaves  $\mathcal{T}_{\text{poly}}^\bullet$ ,  $C^\bullet(\mathcal{S}M)$ ,  $\mathcal{E}^\bullet$ , and  $C_\bullet(\mathcal{S}M)$  are equipped with the canonical action of the sheaf of Lie algebras  $\mathcal{T}_{\text{poly}}^0$  and this action is compatible with the corresponding (DG) algebraic structures. Using this action in [17], chapter 4, we extend the Fedosov differential (2.5) to a differential on the DGLAs (resp. DGLA modules)  $\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)$ ,  $\Omega^\bullet(\mathcal{E}^\bullet)$ ,  $\Omega^\bullet(C_\bullet(\mathcal{S}M))$ , and  $\Omega^\bullet(C^\bullet(\mathcal{S}M))$ .

Using acyclicity of the Fedosov differential (2.5) in positive dimension one constructs in [17] embeddings of DGLAs and DGLA modules<sup>3)</sup>

<sup>2)</sup> The arrow over  $\partial$  in (2.9) means that we use the left derivative with respect to the ‘‘anti-commuting’’ variable  $dy^k$ .

<sup>3)</sup> See [17], Eq. (5.1), p. 81.

$$(2.13) \quad \begin{array}{ccc} T_{\text{poly}}^{\bullet} & \xrightarrow{\lambda_T} & (\Omega^{\bullet}(\mathcal{T}_{\text{poly}}^{\bullet}), D, [\cdot, \cdot]_{\text{SN}}) \\ \downarrow L_{\text{mod}} & & \downarrow L_{\text{mod}} \\ \mathcal{A}^{\bullet} & \xrightarrow{\lambda_{\mathcal{A}}} & (\Omega^{\bullet}(\mathcal{E}^{\bullet}), D), \end{array}$$

$$(2.14) \quad \begin{array}{ccc} (\Omega^{\bullet}(C^{\bullet}(\mathcal{S}M)), D + \partial, [\cdot, \cdot]_{\text{G}}) & \xleftarrow{\lambda_D} & C^{\bullet}(\mathcal{O}_M) \\ \downarrow R_{\text{mod}} & & \downarrow R_{\text{mod}} \\ (\Omega^{\bullet}(C_{\bullet}(\mathcal{S}M)), D + \mathfrak{b}) & \xleftarrow{\lambda_C} & C_{\bullet}(\mathcal{O}_M), \end{array}$$

and shows that these are quasi-isomorphisms of the corresponding complexes.

Furthermore, using Kontsevich’s and Shoikhet’s formality theorems for  $\mathbb{R}^d$  [25], [41] in [17] one constructs the following diagram:

$$(2.15) \quad \begin{array}{ccc} (\Omega^{\bullet}(\mathcal{T}_{\text{poly}}^{\bullet}), D, [\cdot, \cdot]_{\text{SN}}) & \xrightarrow{\mathcal{K}} & (\Omega^{\bullet}(C^{\bullet}(\mathcal{S}M)), D + \partial, [\cdot, \cdot]_{\text{G}}) \\ \downarrow L_{\text{mod}} & & \downarrow R_{\text{mod}} \\ (\Omega^{\bullet}(\mathcal{E}^{\bullet}), D) & \xleftarrow{\mathcal{S}} & (\Omega^{\bullet}(C_{\bullet}(\mathcal{S}M)), D + \mathfrak{b}) \end{array}$$

where  $\mathcal{K}$  is an  $L_{\infty}$  quasi-isomorphism of DGLAs,  $\mathcal{S}$  is a quasi-isomorphism of  $L_{\infty}$ -modules over the DGLA  $(\Omega^{\bullet}(\mathcal{T}_{\text{poly}}^{\bullet}), D, [\cdot, \cdot]_{\text{SN}})$ , and the  $L_{\infty}$ -module structure on  $\Omega^{\bullet}(C_{\bullet}(\mathcal{S}M))$  is obtained by composing the  $L_{\infty}$  quasi-isomorphism  $\mathcal{K}$  with the DGLA modules structure  $R$  (see [17], Eq. (3.5), p. 46, for the definition of  $R$ ).

Diagrams (2.13), (2.14) and (2.15) show that the DGLA module  $C_{\bullet}(\mathcal{O}_M)$  of Hochschild chains of  $\mathcal{O}_M$  is quasi-isomorphic to the graded Lie algebra module  $\mathcal{A}^{\bullet}$  of its cohomology.

**Remark 1.** As in [17] we use adapted versions of Hochschild (co)chains for the algebras  $\mathcal{O}_M$  and  $\text{End}(E)$  of functions and of endomorphisms of a vector bundle  $E$ , respectively. Thus,  $C^{\bullet}(\mathcal{O}_M)$  is the complex of polydifferential operators (see [17], p. 48) satisfying the corresponding normalization condition.  $C^{\bullet}(\text{End}(E))$  is the complex of (normalized) polydifferential operators acting on  $\text{End}(E)$  with coefficients in  $\text{End}(E)$ . Furthermore,  $C_{\bullet}(\mathcal{O}_M)$  is the complex of (normalized) polyjets

$$C_k(\mathcal{O}_M) = \text{Hom}_{\mathcal{O}_M}(C^{k-1}(\mathcal{O}_M), \mathcal{O}_M),$$

and

$$C_k(\text{End}(E)) = \text{Hom}_{\text{End}(E)}(C^{k-1}(\text{End}(E)), \text{End}(E)).$$

We have to warn the reader that the complex  $C^{\bullet}(\mathcal{O}_M)$  (resp.  $C^{\bullet}(\text{End}(E))$ ) does not coincide with the complex of Hochschild cochains of the algebra  $\mathcal{O}_M$  of functions (resp. the

algebra  $\text{End}(E)$  of endomorphisms of  $E$ ). Similar expectation is wrong for Hochschild chains. Instead we have the following inclusions:

$$\begin{aligned} C^\bullet(\mathcal{O}_M) &\subset C_{\text{genuine}}^\bullet(\mathcal{O}_M), & C^\bullet(\text{End}(E)) &\subset C_{\text{genuine}}^\bullet(\text{End}(E)), \\ C_{\bullet}^{\text{genuine}}(\mathcal{O}_M) &\subset C_{\bullet}(\mathcal{O}_M), & C_{\bullet}^{\text{genuine}}(\text{End}(E)) &\subset C_{\bullet}(\text{End}(E)), \end{aligned}$$

where  $C_{\bullet}^{\text{genuine}}$  and  $C_{\text{genuine}}^\bullet$  refer to the original definitions (2.1), (2.2) of Hochschild chains and cochains, respectively.

**Remark 2.** Unlike in [17] we use only normalized Hochschild (co)chains. It is not hard to check that the results we need from [17], [25], and [41] also hold when this normalization condition is imposed.

Let us now recall the construction of paper [16], in which the formality of the DGLA module  $(C^\bullet(\text{End}(E)), C_{\bullet}(\text{End}(E)))$  is proved.

The construction of [16] is based on the use of the following auxiliary sheaf of algebras:

$$(2.16) \quad \mathbb{E}_{\mathcal{S}} = \text{End}(E) \otimes_{\mathcal{O}_M} \mathcal{S}M$$

considered as a sheaf of algebras over  $\mathcal{O}_M$ .

It is shown in [16] that the Fedosov differential (2.5) can be extended to the following differential on  $\Omega^\bullet(\mathbb{E}_{\mathcal{S}})$ :

$$(2.17) \quad D^E = D + [\gamma^E, \cdot], \quad \gamma^E = \Gamma^E + \tilde{\gamma}^E,$$

where  $\Gamma^E$  is a connection form of  $E$  and  $\tilde{\gamma}^E$  is an element in  $\Omega^1(\mathbb{E}_{\mathcal{S}})$  defined by iterating the equation

$$(2.18) \quad \gamma^E = \Gamma^E + \delta^{-1} \left( \nabla \gamma^E + A(\gamma^E) + \frac{1}{2} [\gamma^E, \gamma^E] \right)$$

in degrees in fiber coordinates  $y^i$  of the tangent bundle  $TM$ .

The differential (2.17) naturally extends to the DGLA  $\Omega^\bullet(C^\bullet(\mathbb{E}_{\mathcal{S}}))$  and to the DGLA module  $\Omega^\bullet(C_{\bullet}(\mathbb{E}_{\mathcal{S}}))$ . Namely, on  $\Omega^\bullet(C^\bullet(\mathbb{E}_{\mathcal{S}}))$  the differential  $D^E$  is defined by the formula

$$(2.19) \quad D^E = D + [\partial \gamma^E, \cdot]_{\mathbb{G}},$$

and on  $\Omega^\bullet(C_{\bullet}(\mathbb{E}_{\mathcal{S}}))$  it is defined by the equation

$$(2.20) \quad D^E = D + R_{\partial \gamma^E}.$$

Here,  $\partial$  is the Hochschild coboundary operator and  $R$  denotes the actions of cochains on chains.

Then, generalizing the construction of the maps  $\lambda_D$  and  $\lambda_C$  in (2.14) one gets the following embeddings of DGLAs and their modules:

$$(2.21) \quad \begin{array}{ccc} (\Omega^\bullet(C^\bullet(\mathcal{E}\mathcal{S})), D^E + \partial, [\cdot, \cdot]_G) & \xleftarrow{\lambda_D^E} & C^\bullet(\text{End}(E)) \\ \downarrow R_{\text{mod}} & & \downarrow R_{\text{mod}} \\ (\Omega^\bullet(C_\bullet(\mathcal{E}\mathcal{S})), D^E + \mathfrak{b}) & \xleftarrow{\lambda_C^E} & C_\bullet(\text{End}(E)). \end{array}$$

Similarly to [17], Propositions 7, 13, 15, one can easily show that  $\lambda_D^E$  and  $\lambda_C^E$  are quasi-isomorphisms of the corresponding complexes.

Finally, the DGLA modules

$$(\Omega^\bullet(C^\bullet(\mathcal{E}\mathcal{S})), \Omega^\bullet(C_\bullet(\mathcal{E}\mathcal{S}))) \quad \text{and} \quad (\Omega^\bullet(C^\bullet(\mathcal{S}M)), \Omega^\bullet(C_\bullet(\mathcal{S}M)))$$

are connected in [16] by the following commutative diagram of quasi-isomorphisms of DGLAs and their modules:

$$(2.22) \quad \begin{array}{ccc} (\Omega^\bullet(C^\bullet(\mathcal{S}M)), D + \partial, [\cdot, \cdot]_G) & \xrightarrow{\text{cotr}^{fw}} & (\Omega^\bullet(C^\bullet(\mathcal{E}\mathcal{S})), D^E + \partial, [\cdot, \cdot]_G) \\ \downarrow \text{mod} & & \downarrow \text{mod} \\ (\Omega^\bullet(C_\bullet(\mathcal{S}M)), D + \mathfrak{b}) & \xleftarrow{\text{tr}^{fw}} & (\Omega^\bullet(C_\bullet(\mathcal{E}\mathcal{S})), D^E + \mathfrak{b}), \end{array}$$

where

$$(2.23) \quad \text{cotr}^{fw} = \exp(-[\gamma^E, \cdot]_G) \circ \text{cotr}, \quad \text{tr}^{fw} = \text{tr} \circ \exp(R_{\gamma^E}),$$

and  $\text{tr}$ ,  $\text{cotr}$  are the maps

$$(2.24) \quad \text{tr} : C_\bullet(\mathcal{E}\mathcal{S}) \rightarrow C_\bullet(\mathcal{S}M), \quad \text{cotr} : C^\bullet(\mathcal{S}M) \rightarrow C^\bullet(\mathcal{E}\mathcal{S})$$

defined as in (2.3) and (2.4).

It should be remarked that the element  $\gamma^E$  in (2.23) comprises the connection form of  $E$  (2.17) and hence can be viewed as a section of the sheaf  $\Omega^1(C^{-1}(\mathcal{E}\mathcal{S}))$  only locally. The compositions  $\text{tr}^{fw}$  and  $\text{cotr}^{fw}$  are still well defined due to the fact that we consider normalized Hochschild (co)chains.

Diagrams (2.13), (2.15), (2.21), and (2.22) give us the desired chain of formality quasi-isomorphisms for the DGLA module  $(C^\bullet(\text{End}(E)), C_\bullet(\text{End}(E)))$ .

### 3. The twisting procedure revisited

In this section we will prove some general facts about the twisting by a Maurer-Cartan element. See [17], Section 2.4, in which this procedure is discussed in more details.

Let  $(\mathcal{L}, d_{\mathcal{L}}, [\cdot, \cdot]_{\mathcal{L}})$  ( $\tilde{\mathcal{L}}, d_{\tilde{\mathcal{L}}}, [\cdot, \cdot]_{\tilde{\mathcal{L}}}$ ) and be two DGLAs over  $\mathbb{R}$ . Since we deal with deformation theory questions it is convenient for our purposes to extend the field  $\mathbb{R}$  to the ring  $\mathbb{R}[[\hbar]]$  from the very beginning and consider  $\mathbb{R}[[\hbar]]$ -modules  $\mathcal{L}[[\hbar]]$  and  $\tilde{\mathcal{L}}[[\hbar]]$  with DGLA structures extended from  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  in the obvious way.

By definition  $\alpha$  is a Maurer-Cartan element of the DGLA  $\mathcal{L}[[\hbar]]$  if  $\alpha \in \hbar\mathcal{L}[[\hbar]]$ , it has degree 1 and satisfies the equation

$$(3.1) \quad d_{\mathcal{L}}\alpha + \frac{1}{2}[\alpha, \alpha]_{\mathcal{L}} = 0.$$

The first two conditions can be written concisely as  $\alpha \in \hbar\mathcal{L}^1[[\hbar]]$ .

Notice that,  $\mathfrak{g}(\mathcal{L}) = \hbar\mathcal{L}^0[[\hbar]]$  forms an ordinary (not graded) Lie algebra over  $\mathbb{R}[[\hbar]]$ . Furthermore,  $\mathfrak{g}(\mathcal{L})$  is obviously pronilpotent and hence can be exponentiated to the group

$$(3.2) \quad \mathfrak{G}(\mathcal{L}) = \exp(\hbar\mathcal{L}^0[[\hbar]]).$$

The natural action of this group on  $\mathcal{L}[[\hbar]]$  can be introduced by exponentiating the adjoint action of  $\mathfrak{g}(\mathcal{L})$ .

The action of  $\mathfrak{G}(\mathcal{L})$  on the Maurer-Cartan elements of the DGLA  $\mathcal{L}[[\hbar]]$  is given by the formula

$$(3.3) \quad \exp(\xi)[\alpha] = \alpha + f([\cdot, \xi]_{\mathcal{L}})(d_{\mathcal{L}}\xi + [\alpha, \xi]_{\mathcal{L}}),$$

where  $\alpha$  is a Maurer-Cartan element of  $\mathcal{L}[[\hbar]]$ ,  $\xi \in \hbar\mathcal{L}^0[[\hbar]]$ ,  $f(x)$  is the function

$$f(x) = \frac{e^x - 1}{x}$$

and the expression  $f([\cdot, \xi]_{\mathcal{L}})$  is defined via the Taylor expansion of  $f(x)$  around the point  $x = 0$ .

We call Maurer-Cartan elements equivalent if they lie on the same orbit of the action (3.3). The set of these orbits is called the moduli space of the Maurer-Cartan elements.

We have the following proposition:

**Proposition 2** (W. Goldman and J. Millson, [23]). *If  $f$  is a quasi-isomorphism from the DGLA  $\mathcal{L}$  to the DGLA  $\tilde{\mathcal{L}}$  then the induced map between the moduli spaces of Maurer-Cartan elements of  $\mathcal{L}[[\hbar]]$  and  $\tilde{\mathcal{L}}[[\hbar]]$  is an isomorphism of sets.*

Every Maurer-Cartan element  $\alpha$  of  $\mathcal{L}[[\hbar]]$  can be used to modify the DGLA structure on  $\mathcal{L}[[\hbar]]$ . This modified structure is called [36] the DGLA structure twisted by the Maurer-Cartan  $\alpha$ . The Lie bracket of the twisted DGLA structure is the same and the differential is given by the formula

$$(3.4) \quad d_{\mathcal{L}}^{\alpha} = d_{\mathcal{L}} + [\alpha, \cdot]_{\mathcal{L}}.$$

We will denote the DGLA  $\mathcal{L}[[\hbar]]$  with the bracket  $[\cdot, \cdot]_{\mathcal{L}}$  and the differential  $d_{\mathcal{L}}^{\alpha}$  by  $\mathcal{L}^{\alpha}$ .

Two DGLAs are called quasi-isomorphic if there is a chain of quasi-isomorphisms  $f, f_1, f_2, \dots, f_n$  connecting  $\mathcal{L}$  with  $\tilde{\mathcal{L}}$ :

$$(3.5) \quad \mathcal{L} \xrightarrow{f} \mathcal{L}_1 \xleftarrow{f_1} \mathcal{L}_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \mathcal{L}_n \xleftarrow{f_n} \tilde{\mathcal{L}}.$$

This chain naturally extends to the chain of quasi-isomorphisms of DGLAs over  $\mathbb{R}[[\hbar]]$

$$(3.6) \quad \mathcal{L}[[\hbar]] \xrightarrow{f} \mathcal{L}_1[[\hbar]] \xleftarrow{f_1} \mathcal{L}_2[[\hbar]] \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \mathcal{L}_n[[\hbar]] \xleftarrow{f_n} \tilde{\mathcal{L}}[[\hbar]].$$

We would like to prove that

**Proposition 3.** *For every Maurer-Cartan element  $\alpha$  of  $\mathcal{L}[[\hbar]]$  the chain of quasi-isomorphisms (3.6) can be upgraded to the chain*

$$(3.7) \quad \mathcal{L}^\alpha \xrightarrow{f} \mathcal{L}_1^{\alpha_1} \xleftarrow{\tilde{f}_1} \mathcal{L}_2^{\alpha_2} \xrightarrow{\tilde{f}_2} \dots \xrightarrow{\tilde{f}_{n-1}} \mathcal{L}_n^{\alpha_n} \xleftarrow{\tilde{f}_n} \tilde{\mathcal{L}}^{\tilde{\alpha}},$$

where  $\alpha_i$  (resp.  $\tilde{\alpha}$ ) are Maurer-Cartan elements of  $\mathcal{L}_i[[\hbar]]$  (resp.  $\tilde{\mathcal{L}}[[\hbar]]$ ) and the quasi-isomorphisms  $\tilde{f}_i$  are obtained from  $f_i$  by composing with the action of an element in the group  $\mathfrak{G}(\mathcal{L}_i)$ .

*Proof* runs by induction on  $n$ . We, first, prove the base of the induction ( $n = 1$ ) and then the step follows easily from the statement of the proposition for  $n = 1$ .

We set  $\alpha_1$  to be

$$\alpha_1 = f(\alpha).$$

Since  $f_1$  is a quasi-isomorphism from  $\tilde{\mathcal{L}}$  to  $\mathcal{L}_1$ , by Proposition 2, there exists a Maurer-Cartan element  $\tilde{\alpha} \in \tilde{\mathcal{L}}$  such that  $f_1(\tilde{\alpha})$  is equivalent to  $\alpha_1$ .

Let  $T_1$  be an element of the group  $\mathfrak{G}(\mathcal{L}_1)$  which transforms  $f_1(\tilde{\alpha})$  to  $\alpha_1$ . Thus by setting

$$(3.8) \quad \tilde{f}_1 = T_1 \circ f_1$$

we get the desired chain for  $n = 1$

$$(3.9) \quad \mathcal{L}^\alpha \xrightarrow{f} \mathcal{L}_1^{\alpha_1} \xleftarrow{\tilde{f}_1} \tilde{\mathcal{L}}^{\tilde{\alpha}}$$

and the proposition follows.  $\square$

Chain (3.7) gives us an isomorphism

$$(3.10) \quad I_{\tilde{\alpha}} : H^\bullet(\tilde{\mathcal{L}}^{\tilde{\alpha}}) \xrightarrow{\sim} H^\bullet(\mathcal{L}^\alpha)$$

from the cohomology of the DGLA  $\tilde{\mathcal{L}}^{\tilde{\alpha}}$  to the cohomology of the DGLA  $\mathcal{L}^\alpha$ . This isomorphism depends on choices of Maurer-Cartan elements in the intermediate terms  $\mathcal{L}_i[[\hbar]]$  and the choices of elements from the groups  $\mathfrak{G}(\mathcal{L}_i)$ .

We claim that

**Proposition 4.** *If  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  are DGLAs connected by the chain of quasi-isomorphisms (3.5),  $\alpha$  is a Maurer-Cartan element in  $\hbar\mathcal{L}[[\hbar]]$  and*

$$(3.11) \quad \mathcal{L}^\alpha \xrightarrow{f} \mathcal{L}_1^{\alpha'_1} \xleftarrow{\tilde{f}'_1} \mathcal{L}_2^{\alpha'_2} \xrightarrow{\tilde{f}'_2} \dots \xrightarrow{\tilde{f}'_{n-1}} \mathcal{L}_n^{\alpha'_n} \xleftarrow{\tilde{f}'_n} \tilde{\mathcal{L}}^{\tilde{\alpha}'},$$

*is another chain of quasi-isomorphism obtained according to Proposition 3 then there exists an element  $T_{\tilde{\mathcal{L}}}$  of the group  $\mathfrak{G}(\tilde{\mathcal{L}})$  such that*

$$(3.12) \quad T_{\tilde{\mathcal{L}}}(\tilde{\alpha}) = \tilde{\alpha}',$$

and

$$(3.13) \quad I_{\tilde{\alpha}} = I_{\tilde{\alpha}'} \circ T_{\tilde{\mathcal{L}}}.$$

Before proving the proposition let us consider the case  $n = 0$  with a slight modification. More precisely, we consider a quasi-isomorphism  $g = f_0$  from the DGLA  $\tilde{\mathcal{L}}$  to the DGLA  $\mathcal{L}$

$$(3.14) \quad \mathcal{L} \xleftarrow{g} \tilde{\mathcal{L}}$$

and suppose that  $\alpha$  and  $\alpha'$  are two equivalent Maurer-Cartan elements of  $\mathcal{L}[[\hbar]]$ .

Due to Proposition 2 there exist Maurer-Cartan elements  $\tilde{\alpha}$  and  $\tilde{\alpha}'$  in  $\tilde{\mathcal{L}}[[\hbar]]$  and elements  $T$  and  $T'$  of the group  $\mathfrak{G}(\mathcal{L})$  such that

$$T(g(\tilde{\alpha})) = \alpha, \quad T'(g(\tilde{\alpha}')) = \alpha'.$$

Hence, by setting

$$(3.15) \quad \tilde{g} = T \circ g, \quad \tilde{g}' = T' \circ g$$

we get the following pair of quasi-isomorphisms of twisted DGLAs:

$$(3.16) \quad \mathcal{L}^\alpha \xleftarrow{\tilde{g}} \tilde{\mathcal{L}}^{\tilde{\alpha}},$$

$$(3.17) \quad \mathcal{L}^{\alpha'} \xleftarrow{\tilde{g}'} \tilde{\mathcal{L}}^{\tilde{\alpha}'}$$

Let us prove the following auxiliary statement:

**Lemma 1.** *If  $\alpha$  and  $\alpha'$  are Maurer-Cartan elements of  $\mathcal{L}[[\hbar]]$  connected by the action of an element  $T_{\mathcal{L}} \in \mathfrak{G}(\mathcal{L})$*

$$T_{\mathcal{L}}(\alpha) = \alpha',$$

*then there exists an element  $T_{\tilde{\mathcal{L}}}$  of the group  $\mathfrak{G}(\tilde{\mathcal{L}})$  such that*

$$T_{\tilde{\mathcal{L}}}(\tilde{\alpha}) = \tilde{\alpha}'$$

and the diagram of DGLAs

$$(3.18) \quad \begin{array}{ccc} \mathcal{L}^\alpha & \xleftarrow{\tilde{g}} & \tilde{\mathcal{L}}^{\tilde{\alpha}} \\ \downarrow T_{\mathcal{L}} & & \downarrow T_{\tilde{\mathcal{L}}} \\ \mathcal{L}^{\alpha'} & \xleftarrow{\tilde{g}'} & \tilde{\mathcal{L}}^{\tilde{\alpha}'} \end{array}$$

commutes up to homotopy.

*Proof.* Since the Maurer-Cartan element  $g(\tilde{\alpha})$  and  $g(\tilde{\alpha}')$  are equivalent, then so are the Maurer-Cartan elements  $\tilde{\alpha}$  and  $\tilde{\alpha}'$ . Hence, there exists an element  $T_0 \in \mathfrak{G}(\tilde{\mathcal{L}})$  such that

$$T_0(\tilde{\alpha}) = \tilde{\alpha}'.$$

The following diagram shows the relations between various Maurer-Cartan elements in question:

$$(3.19) \quad \begin{array}{ccccc} \alpha & \xleftarrow{T} & g(\tilde{\alpha}) & \xleftarrow{g} & \tilde{\alpha} \\ \downarrow T_{\mathcal{L}} & & \downarrow g(T_0) & & \downarrow T_0 \\ \alpha' & \xleftarrow{T'} & g(\tilde{\alpha}') & \xleftarrow{g} & \tilde{\alpha}'. \end{array}$$

From this diagram we see that

$$T_{\mathcal{L}} T[g(\tilde{\alpha})] = T' g(T_0)[g(\tilde{\alpha})],$$

or equivalently

$$(3.20) \quad g(T_0)^{-1} (T')^{-1} T_{\mathcal{L}} T[g(\tilde{\alpha})] = [g(\tilde{\alpha})].$$

Therefore the element  $g(T_0)^{-1} (T')^{-1} T_{\mathcal{L}} T$  belongs to the subgroup  $\mathfrak{G}(\mathcal{L}, g(\tilde{\alpha})) \subset \mathfrak{G}(\mathcal{L})$  of elements preserving  $g(\tilde{\alpha})$ .

It is not hard to show that the subgroup  $\mathfrak{G}(\mathcal{L}, g(\tilde{\alpha}))$  is

$$(3.21) \quad \mathfrak{G}(\mathcal{L}, g(\tilde{\alpha})) = \exp(\hbar \mathcal{L}^0[[\hbar]]) \cap \ker(d_{\mathcal{L}} + [g(\tilde{\alpha}), \ ]_{\mathcal{L}}).$$

In other words, there exists a  $d_{\mathcal{L}} + [g(\tilde{\alpha}), \ ]_{\mathcal{L}}$ -closed element  $\xi \in \hbar \mathcal{L}^0[[\hbar]]$  such that

$$g(T_0)^{-1} (T')^{-1} T_{\mathcal{L}} T = \exp(\xi).$$

Since the map  $g$  from  $\tilde{\mathcal{L}}^{\tilde{\alpha}}$  to  $\mathcal{L}^{g(\tilde{\alpha})}$  is a quasi-isomorphism, there exists an element  $\psi \in \hbar \tilde{\mathcal{L}}^0[[\hbar]]$  such that

$$d_{\tilde{\mathcal{L}}} \psi + [\tilde{\alpha}, \psi]_{\tilde{\mathcal{L}}} = 0,$$

and the difference  $\xi - g(\psi)$  is  $d_{\mathcal{L}} + [g(\tilde{\alpha}), \ ]_{\mathcal{L}}$ -exact. Therefore the elements

$$T_{\mathcal{L}} T$$



and

$$T'g(T_0 \exp(\psi))$$

induce homotopic maps from the DGLA  $\mathcal{L}^{g(\tilde{\alpha})}$  to the DGLA  $\mathcal{L}^{\alpha'}$ .

Thus,  $T_{\tilde{\mathcal{L}}} = T_0 \exp(\psi)$  is the desired element of the group  $\mathfrak{G}(\tilde{\mathcal{L}})$  which makes Diagram (3.18) commutative up to homotopy.

The lemma is proved.  $\square$

*Proof of Proposition 4.* The proof runs by induction on  $n$  and the base of the induction  $n = 0$  follows immediately from Lemma 1.

If the statement is proved for  $n = 2k$  then the statement for  $n = 2k + 1$  also follows from Lemma 1.

Since the source of the map  $f_{2k}$  is  $\mathcal{L}_{2k}$  the case  $n = 2k$  follows immediately from the case  $n = 2k - 1$  and this concludes the proof of the proposition.  $\square$

For a DGLA module  $\mathcal{M}$  over a DGLA  $\mathcal{L}$  the direct sum  $\mathcal{L} \oplus \mathcal{M}$  carries the natural structure of a DGLA. Namely, this DGLA is the semi-direct product of  $\mathcal{L}$  and  $\mathcal{M}$  where  $\mathcal{M}$  is viewed as a DGLA with the zero bracket. Morphisms between two such semi-direct products  $\mathcal{L} \oplus \mathcal{M}$  and  $\tilde{\mathcal{L}} \oplus \tilde{\mathcal{M}}$  correspond to morphisms between the DGLA modules  $(\mathcal{L}, \mathcal{M})$  and  $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}})$ . Furthermore, twisting the DGLA structure on  $\mathcal{L}[[\hbar]] \oplus \mathcal{M}[[\hbar]]$  by a Maurer-Cartan element  $\alpha \in \hbar \mathcal{L}[[\hbar]]$  gives us the semi-direct product  $\mathcal{L}^\alpha \oplus \mathcal{M}^\alpha$  corresponding to the DGLA module  $\mathcal{M}^\alpha$  with the differential twisted by the action of the Maurer-Cartan element  $\alpha$ .

This observation allows us to generalize Propositions 3 and 4 to a chain of quasi-isomorphisms of DGLA modules:

$$(3.22) \quad (\mathcal{L}, \mathcal{M}) \xrightarrow{h} (\mathcal{L}_1, \mathcal{M}_1) \xleftarrow{h_1} (\mathcal{L}_2, \mathcal{M}_2) \xrightarrow{h_2} \dots \xrightarrow{h_{n-1}} (\mathcal{L}_n, \mathcal{M}_n) \xleftarrow{h_n} (\tilde{\mathcal{L}}, \tilde{\mathcal{M}}).$$

Namely,

**Proposition 5.** *For every Maurer-Cartan element  $\alpha$  of  $\mathcal{L}[[\hbar]]$  the chain of quasi-isomorphisms (3.22) can be upgraded to the chain*

$$(3.23) \quad (\mathcal{L}^\alpha, \mathcal{M}^\alpha) \xrightarrow{h} (\mathcal{L}_1^{\alpha_1}, \mathcal{M}_1^{\alpha_1}) \xleftarrow{\tilde{h}_1} (\mathcal{L}_2^{\alpha_2}, \mathcal{M}_2^{\alpha_2}) \\ \xrightarrow{\tilde{h}_2} \dots \xrightarrow{\tilde{h}_{n-1}} (\mathcal{L}_n^{\alpha_n}, \mathcal{M}_n^{\alpha_n}) \xleftarrow{\tilde{h}_n} (\tilde{\mathcal{L}}^{\tilde{\alpha}}, \tilde{\mathcal{M}}^{\tilde{\alpha}}),$$

where  $\alpha_i$  (resp.  $\tilde{\alpha}$ ) are Maurer-Cartan elements of  $\mathcal{L}_i[[\hbar]]$  (resp.  $\tilde{\mathcal{L}}[[\hbar]]$ ) and the quasi-isomorphisms  $\tilde{h}_i$  are obtained from  $h_i$  by composing with the action of an element in the group  $\mathfrak{G}(\mathcal{L}_i)$ .  $\square$

Chain (3.23) gives us an isomorphism

$$(3.24) \quad J_{\tilde{\alpha}} : H^\bullet(\tilde{\mathcal{M}}^{\tilde{\alpha}}) \xrightarrow{\sim} H^\bullet(\mathcal{M}^\alpha)$$

from the cohomology of the DGLA module  $\tilde{\mathcal{M}}^{\tilde{\alpha}}$  to the cohomology of the DGLA module  $\mathcal{M}^\alpha$ . This isomorphism depends on choices of Maurer-Cartan elements in the DGLAs  $\mathcal{L}_i[[\hbar]]$  and the choices of elements from the groups  $\mathfrak{G}(\mathcal{L}_i)$ .

We claim that

**Proposition 6.** *If  $(\mathcal{L}, \mathcal{M})$  and  $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}})$  are DGLA modules connected by the chain of quasi-isomorphisms (3.22),  $\alpha$  is a Maurer-Cartan element in  $\hbar\mathcal{L}[[\hbar]]$  and*

$$(3.25) \quad (\mathcal{L}^\alpha, \mathcal{M}^\alpha) \xrightarrow{h} (\mathcal{L}_1^{\alpha'_1}, \mathcal{M}_1^{\alpha'_1}) \xleftarrow{\tilde{h}'_1} (\mathcal{L}_2^{\alpha'_2}, \mathcal{M}_2^{\alpha'_2}) \\ \xrightarrow{\tilde{h}'_2} \dots \xrightarrow{\tilde{h}'_{n-1}} (\mathcal{L}_n^{\alpha'_n}, \mathcal{M}_n^{\alpha'_n}) \xleftarrow{\tilde{h}'_n} (\tilde{\mathcal{L}}^{\tilde{\alpha}'}, \tilde{\mathcal{M}}^{\tilde{\alpha}'})$$

is another chain of quasi-isomorphism obtained according to Proposition 5 then there exists an element  $T_{\tilde{\mathcal{L}}}$  of the group  $\mathfrak{G}(\tilde{\mathcal{L}})$  such that

$$(3.26) \quad T_{\tilde{\mathcal{L}}}(\tilde{\alpha}) = \tilde{\alpha}'$$

and

$$(3.27) \quad J_{\tilde{\alpha}} = J_{\tilde{\alpha}'} \circ T_{\tilde{\mathcal{L}}}. \quad \square$$

#### 4. Trace density, quantum and classical index densities

In this section we recall the construction of the trace density map (1.4) which gives the quantum index density (1.5). We also construct the classical index density (1.10) using the chain (2.13), (2.15), (2.21), (2.22) of formality quasi-isomorphisms for  $C_\bullet(\text{End}(E))$ .

Let, as above,  $\mathcal{O}_M^{\hbar} = (\mathcal{O}_M[[\hbar]], *)$  be a deformation quantization algebra of the Poisson manifold  $(M, \pi_1)$  and  $\pi$  (1.1) be a representative of Kontsevich's class of  $\mathcal{O}_M^{\hbar}$ .

It can be shown that every star-product  $*$  is equivalent to the so-called natural star-product [24]. These are the star-products

$$a * b = ab + \sum_{k=1}^{\infty} \hbar^k B_k(a, b)$$

for which the bidifferential operators  $B_k$  satisfy the following condition:

**Condition 1.** *For all  $k \geq 1$  the bidifferential operator  $B_k$  has the order at most  $k$  in each argument.*

In this paper we tacitly assume that the above condition holds for the star-product  $*$ .

Using the map  $\lambda_T$  from (2.13) we lift  $\pi$  to the Maurer-Cartan element  $\lambda_T(\pi)$  in the DGLA  $\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]]$ . This element allows us to extend the differential  $D$  (2.5) on  $\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]]$  to

$$(4.1) \quad D + [\lambda_T(\pi), ]_{\text{SN}} : \Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]] \rightarrow (\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]])[1],$$

where  $[1]$  denotes the shift of the total degree by 1.

Similarly, we extend the differential  $D$  (2.5) on  $\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]]$  to

$$(4.2) \quad D + L_{\lambda_T(\pi)} : \Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]] \rightarrow (\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]])[1].$$

Notice that, the star-product  $*$  in  $\mathcal{O}_M^\hbar$  can be rewritten in the form

$$(4.3) \quad a * b = ab + \Pi(a, b), \quad a, b \in \mathcal{O}_M[[\hbar]],$$

where  $\Pi \in \hbar C^1(\mathcal{O}_M)[[\hbar]]$  can be viewed as a Maurer-Cartan element of  $C^\bullet(\mathcal{O}_M)[[\hbar]]$ .

Applying the map  $\lambda_D$  (2.14) to  $\Pi$  we get a  $D$ -flat Maurer-Cartan element in  $\hbar\Gamma(M, C^\bullet(\mathcal{S}M))[[\hbar]]$  and hence a new product in  $\mathcal{S}M[[\hbar]]$ :

$$(4.4) \quad a \diamond b = ab + \lambda_D(\Pi)(a, b), \quad a, b \in \Gamma(M, \mathcal{S}M)[[\hbar]].$$

Condition 1 implies that the product  $\diamond$  is compatible with the following filtration on  $\mathcal{S}M[[\hbar]]$ :

$$(4.5) \quad \dots \subset F^k \mathcal{S}M[[\hbar]] \subset F^{k-1} \mathcal{S}M[[\hbar]] \subset \dots \subset F^0 \mathcal{S}M[[\hbar]] = \mathcal{S}M[[\hbar]],$$

where the local sections of  $F^k \mathcal{S}M[[\hbar]]$  are the following formal power series:

$$(4.6) \quad \Gamma(F^k \mathcal{S}M[[\hbar]]) = \left\{ \sum_{2p+m \geq k} a_{p; i_1 \dots i_m}(x) \hbar^p y^{i_1} \dots y^{i_m} \right\}.$$

Using the product  $\diamond$  we extend the original differentials  $D + \partial$  and  $D + \flat$  on  $\Omega^\bullet(C^\bullet(\mathcal{S}M))[[\hbar]]$  and  $\Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]]$  to

$$(4.7) \quad D + \partial_\diamond : \Omega^\bullet(C^\bullet(\mathcal{S}M))[[\hbar]] \rightarrow (\Omega^\bullet(C^\bullet(\mathcal{S}M))[[\hbar]])[1],$$

and

$$(4.8) \quad D + \flat_\diamond : \Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]] \rightarrow (\Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]])[1],$$

respectively.

Here  $\partial_\diamond$  (resp.  $\flat_\diamond$ ) is the Hochschild coboundary (resp. boundary) operator corresponding to (4.4), and  $[1]$  as above denotes the shift of the total degree by 1.

Next, following the lines of [17], section 5.3, we can construct the following chain of  $(L_\infty)$  quasi-isomorphisms of DGLA modules:

$$(4.9) \quad \begin{array}{ccccccc} T_{\text{poly}}^\bullet[[\hbar]] & \xrightarrow{\lambda_T} & \Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]] & \xrightarrow{\tilde{\mathcal{K}}} & \Omega^\bullet(C^\bullet(\mathcal{S}M))[[\hbar]] & \xleftarrow{\lambda_D} & C^\bullet(\mathcal{O}_M^\hbar) \\ \downarrow L_{\text{mod}} & & \downarrow L_{\text{mod}} & & \downarrow R_{\text{mod}} & & \downarrow R_{\text{mod}} \\ \mathcal{A}^\bullet[[\hbar]] & \xrightarrow{\lambda_{\mathcal{A}}} & \Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]] & \xleftarrow{\tilde{\mathcal{S}}} & \Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]] & \xleftarrow{\lambda_C} & C_\bullet(\mathcal{O}_M^\hbar), \end{array}$$

where  $T_{\text{poly}}^\bullet[[\hbar]]$  carries the Lichnerowicz differential  $[\pi, ]_{\text{SN}}$ ,  $\mathcal{A}^\bullet[[\hbar]]$  carries the differential  $L_\pi$ , while  $\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)[[\hbar]]$ ,  $\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]]$ ,  $\Omega^\bullet(C^\bullet(\mathcal{S}M))[[\hbar]]$ ,  $\Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]]$  carry the differentials (4.1), (4.2), (4.7), (4.8), respectively.

The maps  $\lambda_T$  and  $\lambda_D$ ,  $\lambda_{\mathcal{A}}$ ,  $\lambda_C$  are genuine morphisms of DGLAs and their modules as in Equations (2.13) and (2.14).  $\tilde{\mathcal{K}}$  is an  $L_\infty$ -quasi-isomorphism of the DGLAs and  $\tilde{\mathcal{S}}$  is a quasi-isomorphism of the corresponding  $L_\infty$ -modules.  $\tilde{\mathcal{K}}$  and  $\tilde{\mathcal{S}}$  are obtained from  $\mathcal{K}$  and  $\mathcal{S}$  in (2.15), respectively, in two steps. First, we twist<sup>4)</sup>  $\mathcal{K}$  and  $\mathcal{S}$  by the Maurer-Cartan element  $\lambda_T(\pi)$ . Second, we adjust them by the action of an element  $T$  of the prounipotent group

$$(4.10) \quad \mathfrak{G}(\Omega^\bullet(C^\bullet(\mathcal{S}M))) = \exp[\mathfrak{g}(\Omega^\bullet(C^\bullet(\mathcal{S}M)))]$$

corresponding to the Lie algebra

$$\mathfrak{g}(\Omega^\bullet(C^\bullet(\mathcal{S}M))) = \hbar\Gamma(M, C^0(\mathcal{S}M))[[\hbar]] \oplus \hbar\Omega^1(C^{-1}(\mathcal{S}M))[[\hbar]].$$

The element  $T \in \mathfrak{G}(\Omega^\bullet(C^\bullet(\mathcal{S}M)))$  is defined as an element which transforms the Maurer-Cartan element

$$(4.11) \quad \sum_{m=1}^{\infty} \frac{1}{m!} \mathcal{H}_m(\lambda_T(\pi), \dots, \lambda_T(\pi))$$

to the Maurer-Cartan element  $\lambda_D(\Pi)$ .

The desired trace density map (1.4) is defined as the composition

$$(4.12) \quad \text{trd} = [H^\bullet(\lambda_{\mathcal{A}})]^{-1} \circ H^\bullet(\tilde{\mathcal{S}}_0) \circ H^\bullet(\lambda_C)|_{HH_0(\mathcal{O}_M^\hbar)},$$

where  $H^\bullet$  denotes the cohomology functor, and  $\tilde{\mathcal{S}}_0$  is the structure map of the zeroth level of the morphism  $\tilde{\mathcal{S}}$  in (4.9).

We have to mention that the construction of the map (4.12) depends on the choice of the element  $T$  in the group (4.10) which transforms the Maurer-Cartan element (4.11) to  $\lambda_D(\Pi)$ , where  $\Pi$  is defined in (4.3). Proposition 6 implies that altering the element  $T$  changes the trace density by the action of an automorphism of  $\mathcal{O}_M^\hbar$  which is trivial modulo

<sup>4)</sup> See [17], section 2.4 about the twisting procedure.

$\hbar$ . In the symplectic case all such automorphisms are inner, while for a general Poisson manifold there may be non-trivial outer automorphisms. Fortunately, we have the following proposition:

**Proposition 7.** *The composition (1.5) of the trace density (4.12) and the map (1.3) is independent of the choice of  $T$  in the construction of the trace density map.*

*Proof.* Let  $\text{ind}$  and  $\widetilde{\text{ind}}$  be two quantum index densities corresponding to different choices of the element  $T$  from the group  $\mathfrak{G}(\Omega^\bullet(C^\bullet(\mathcal{S}M)))$  (4.10).

Due to Proposition 6 there is an automorphism  $\tau$  of  $\mathcal{O}_M^\hbar$  such that

$$(4.13) \quad \tau = 1 \pmod{\hbar},$$

and for every  $\Xi \in K_0(\mathcal{O}_M^\hbar)$

$$(4.14) \quad \widetilde{\text{ind}}(\Xi) = \text{ind}(\hat{\tau}(\Xi)),$$

where  $\hat{\tau}$  denotes the action of  $\tau$  on  $K_0(\mathcal{O}_M^\hbar)$ .

But due to Proposition 1 the image  $\text{ind}(\Xi)$  depends only on the principal symbol  $\sigma(\Xi)$ . Therefore, since  $\tau$  does not change the principal symbol,  $\widetilde{\text{ind}}(\Xi) = \text{ind}(\Xi)$ .  $\square$

Let us now define the classical index density  $\text{ind}_c$  (1.10) which is a map from the  $K$ -theory of  $\mathcal{O}_M$  to the zeroth Poisson homology of  $\pi$  (1.1).

The well known construction of R. G. Swan [42] gives us the injection<sup>5)</sup>

$$(4.15) \quad \varepsilon : K_0(\mathcal{O}_M) \hookrightarrow K^0(M)$$

from the  $K$ -theory of  $\mathcal{O}_M$  to the  $K$ -theory of the manifold  $M$ . Thus, it suffices to define the map  $\text{ind}_c$  on smooth real vector bundles.

For this, we introduce a smooth real vector bundle  $E$  over  $M$  and denote by  $\text{End}(E)$  the algebra of endomorphisms of  $E$ .

Due to Proposition 2 and the formality of the DGLA  $C^\bullet(\text{End}(E))$  [4], [16] a Maurer-Cartan element  $\pi$  (1.1) produces a Maurer-Cartan element  $\Pi_E$  of the DGLA  $\hbar C^\bullet(\text{End}(E))[[\hbar]]$ . This element  $\Pi_E$  gives us the new associative product

$$(4.16) \quad \begin{aligned} a *_E b &= ab + \Pi_E(a, b), \\ a, b &\in \text{End}(E)[[\hbar]] \end{aligned}$$

on the algebra  $\text{End}(E)[[\hbar]]$ .

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<sup>5)</sup> For a compact manifold  $M$  this map is a bijection.

Due to Proposition 5 the chain of quasi-isomorphisms (2.13), (2.15), (2.21), and (2.22) for the DGLA module  $(C^\bullet(\text{End}(E)), C_\bullet(\text{End}(E)))$  can be upgraded<sup>6)</sup> to the chain of quasi-isomorphisms connecting the DGLA module  $(C^\bullet(\text{End}(E))^{\Pi_E}, C_\bullet(\text{End}(E))^{\Pi_E})$  to the DGLA module  $(T_{\text{poly}}^\bullet[[\hbar]], \mathcal{A}^\bullet[[\hbar]])$  where  $T_{\text{poly}}^\bullet[[\hbar]]$  carries the differential  $[\pi, \ ]_{\text{SN}}$  and  $\mathcal{A}^\bullet[[\hbar]]$  carries the differential  $L_\pi$ .

This chain of quasi-isomorphisms gives us the isomorphism

$$J_E : H_\bullet(C_\bullet(\text{End}(E))^{\Pi_E}) \rightarrow H_\bullet(\mathcal{A}^\bullet[[\hbar]], L_\pi)$$

from the homology of the DGLA module  $C_\bullet(\text{End}(E))^{\Pi_E}$  to the homology of the chain complex  $(\mathcal{A}^\bullet, L_\pi)$ .

Since the complex  $C_\bullet(\text{End}(E))^{\Pi_E}$  is nothing but the Hochschild chain complex for the algebra  $\text{End}(E)[[\hbar]]$  with the product (4.16), specifying the map  $J_E$  for  $\bullet = 0$  we get the isomorphism

$$(4.17) \quad \text{trd}_E : HH_0(\text{End}(E)[[\hbar]], *_E) \rightarrow HP_0(M, \pi),$$

from the zeroth Hochschild homology of the algebra  $(\text{End}(E)[[\hbar]], *_E)$  to the zeroth Poisson homology of  $\pi$ .

Using the map (4.17) we define the index density map by the equation

$$(4.18) \quad \text{ind}_c([E]) = \text{trd}_E([1_E]),$$

where  $[1_E]$  is the class in  $HH_0(\text{End}(E)[[\hbar]], *_E)$  represented by the identity endomorphism of  $E$ .

Since  $C^\bullet(\text{End}(E))$  is the normalized Hochschild complex, the group  $\mathfrak{G}(C^\bullet(\text{End}(E)))$  acts trivially on the  $1_E$ . Thus, Proposition 6 implies that the map  $\text{ind}_c$  does not depend on the choices involved in the construction of the isomorphism  $\text{trd}_E$ .

The construction of the chain of the formality quasi-isomorphisms (2.13), (2.15), (2.21), and (2.22) for the DGLA module  $(C^\bullet(\text{End}(E)), C_\bullet(\text{End}(E)))$  involves the choices of the connections on the tangent bundle  $TM$  and on the bundle  $E$  over  $M$ . To show that the map  $\text{ind}_c$  (4.18) is indeed well defined we need to show that

**Proposition 8.** *The image  $\text{ind}_c([E])$  does not depend on the choice of the connections  $\nabla$  and  $\nabla^E$  on bundles  $TM$  and  $E$ .*

*Proof.* By changing the connections on  $TM$  and  $E$  we change the Fedosov differentials  $D$  (2.5) and  $D^E$  (2.17). This means that we twist the DGLAs  $\Omega^\bullet(\mathcal{T}_{\text{poly}}^\bullet)$ ,  $\Omega^\bullet(C^\bullet(\mathcal{S}M))$ , and  $\Omega^\bullet(C^\bullet(E\mathcal{S}))$  by Maurer-Cartan elements. Thus, if we show that these Maurer-Cartan

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<sup>6)</sup> Here we also use the fact that every  $L_\infty$ -quasi-isomorphism can be replaced by a pair of genuine (not  $L_\infty$ ) quasi-isomorphisms.

elements are trivial (equivalent to zero), the question of independence on the connections could be easily reduced to the application of Proposition 6.

Since the differential (2.5) can be viewed as a particular case of the differential (2.17) it suffices to analyze the differential  $D^E$  (2.17).

Changing the Fedosov differential (2.17) on the DGLA  $(\Omega^\bullet(C^\bullet(\mathcal{E}\mathcal{S})), D^E + \partial, [, ]_G)$  and the DGLA module  $(\Omega^\bullet(C_\bullet(\mathcal{E}\mathcal{S})), D^E + \flat)$  corresponds to twisting the DGLA structures by the Maurer-Cartan element

$$(4.19) \quad B^E \in \Omega^1(C^0(\mathcal{E}\mathcal{S}))$$

satisfying the condition

$$(4.20) \quad \partial B^E = 0.$$

Condition (4.20) implies that  $B^E$  is a Maurer-Cartan element of the DGLA

$$(4.21) \quad \begin{aligned} \Omega^0(C^\bullet(\mathcal{E}\mathcal{S})) \cap \ker \partial &\xrightarrow{D^E} \Omega^1(C^\bullet(\mathcal{E}\mathcal{S})) \cap \ker \partial \\ &\xrightarrow{D^E} \Omega^2(C^\bullet(\mathcal{E}\mathcal{S})) \cap \ker \partial \xrightarrow{D^E} \dots \end{aligned}$$

Thus, in virtue of Proposition 2, it suffices to show that the DGLA (4.21) is acyclic in positive exterior degree.

Let  $P \in \Omega^{\geq 1}(C^\bullet(\mathcal{E}\mathcal{S})) \cap \ker \partial$  and

$$(4.22) \quad D^E P = 0.$$

Let us show that an element  $S \in \Omega^\bullet(C^\bullet(\mathcal{E}\mathcal{S}))$  satisfying the equations

$$(4.23) \quad D^E S = P,$$

$$(4.24) \quad \partial S = 0,$$

can be constructed by iterating the following equation

$$(4.25) \quad S = -\delta^{-1}P + \delta^{-1}(\nabla S + A(S) + [\partial\gamma^E, S]_G)$$

in degrees in the fiber coordinates  $y^i$ .

Unfolding the definition of  $D^E$  (2.5), (2.17) we rewrite the difference

$$(4.26) \quad \Lambda = D^E S - P$$

in the form

$$(4.27) \quad \Lambda = \nabla S - \delta S + A(S) + [\partial\gamma^E, S]_G - P.$$

Equation (4.25) implies that  $\delta^{-1}S = 0$  and  $\chi(S) = 0$ , where  $\chi$  is defined in Equation (2.12).

Therefore, applying Equation (2.11) to  $S$  we get

$$S = \delta^{-1}\delta S.$$

Hence,

$$(4.28) \quad \delta^{-1}\Lambda = 0.$$

On the other hand, Equation (4.22) implies that

$$D^E\Lambda = 0,$$

which is equivalent to

$$(4.29) \quad \delta\Lambda = \nabla\Lambda + A(\Lambda) + [\partial\gamma^E, \Lambda]_G.$$

Thus applying (2.11) to  $\Lambda$ , using Equation (4.28) and the fact that  $\Lambda \in \Omega^{\geq 1}(C^\bullet(E\mathcal{S}))$  we get

$$(4.30) \quad \Lambda = \delta^{-1}(\nabla\Lambda + A(\Lambda) + [\partial\gamma^E, \Lambda]_G).$$

The latter equation has the unique vanishing solution since  $\delta^{-1}$  raises the degree in the fiber coordinates  $y^i$ .

The operators  $\delta^{-1}$  (2.9) and  $\nabla$  (2.6) anticommute with  $\partial$ . Furthermore  $\partial A = 0$  by definition of the form  $A$  (2.8). Hence, Equation (4.24) follows from the definition of  $S$  (4.25).

This concludes the proof of the proposition.  $\square$

### 5. The algebraic index theorem

Let us now formulate and prove the main result of this paper:

**Theorem 1.** *Let  $\mathcal{O}_M^{\hbar}$  be a deformation quantization algebra of the Poisson manifold  $(M, \pi_1)$  and let  $\pi$  (1.1) be a representative of Kontsevich's class of  $\mathcal{O}_M^{\hbar}$ . If  $\text{ind}$  is the quantum index density (1.5),  $\text{ind}_c$  is the classical index density (4.18) and  $\sigma$  is principal symbol map (1.6) then the diagram*

$$(5.1) \quad \begin{array}{ccc} K_0(\mathcal{O}_M^{\hbar}) & \xrightarrow{\text{ind}} & HP_0(M, \hbar\pi)[[\hbar]] \\ & \searrow \sigma & \nearrow \text{ind}_c \\ & & K_0(\mathcal{O}_M) \end{array}$$

commutes.



The rest of the section is devoted to the proof of the theorem.

Let  $N$  be an arbitrary natural number and  $P$  be an arbitrary idempotent in the algebra  $\text{Mat}_N(\mathcal{O}_M^{\hbar})$  of  $N \times N$  matrices over  $\mathcal{O}_M^{\hbar}$ . Let  $q$  be the principal symbol of  $P$

$$q = P|_{\hbar=0}.$$

Our purpose is to show that the index  $\text{ind}([P])$  of the class  $[P]$  represented by  $P$  coincides with the image  $\text{ind}_c([E])$ , where  $E$  is the vector bundle defined by  $q$ .

Notice that  $\text{Mat}_N(\mathcal{S}M)$  is  $\mathcal{S}M \otimes \text{End}(I_N)$ , where  $I_N$  denotes the trivial bundle of rank  $N$ . As in [11] we would like to modify the connection (2.6) which is used in the construction of the Fedosov differential (2.5). More precisely, we replace  $\nabla$  (2.6) by

$$(5.2) \quad \nabla^q = \nabla + [\Gamma^q, \cdot] : \mathcal{S}M \otimes \text{End}(I_N) \rightarrow \Omega^1(\mathcal{S}M \otimes \text{End}(I_N)),$$

where

$$\Gamma^q = q(dq) - (dq)q.$$

This connection is distinguished by the following property:

$$(5.3) \quad \nabla^q(q) = 0.$$

In general the connection  $\nabla^q - \delta + A$  is no longer flat. To cure this problem we try to find the flat connection within the framework of the following ansatz:

$$(5.4) \quad D^q = D + [B^q, \cdot]_{\diamond} : \mathcal{S}M \otimes \text{End}(I_N)[[\hbar]] \rightarrow \Omega^1(\mathcal{S}M \otimes \text{End}(I_N))[[\hbar]],$$

where  $B^q \in \Omega^1(\mathcal{S}M \otimes \text{End}(I_N))[[\hbar]]$ ,

$$B^q|_{\hbar=0} = \Gamma^q,$$

and  $[\cdot, \cdot]_{\diamond}$  is the commutator of sections of  $\text{Mat}_N(\mathcal{S}M)[[\hbar]]$ , where the algebra  $\mathcal{S}M[[\hbar]]$  is considered with the product (4.4).

The following proposition shows that the desired section  $B^q$  does exist:

**Proposition 9.** *Iterating the equation*

$$(5.5) \quad B^q = \Gamma^q + \delta^{-1} \left( \nabla B^q + A(B^q) + \frac{1}{2} [B^q, B^q]_{\diamond} \right)$$

one gets an element  $B^q \in \Omega^1(\text{Mat}_N(\mathcal{S}M))[[\hbar]]$  satisfying the equation

$$(5.6) \quad DB^q + \frac{1}{2} [B^q, B^q]_{\diamond} = 0.$$

*Proof.* First, we mention that the process of the recursion in (5.5) converges because

$$(5.7) \quad \delta^{-1}[F^k \Omega^\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]]))] \subset F^{k+1} \Omega^\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]])),$$

where  $\delta^{-1}$  is defined in (2.9) and  $F^\bullet$  is the filtration on  $\mathcal{S}M[[\hbar]]$  defined in (4.6).

Second, since  $\Gamma^q$  does not depend on fiber coordinates  $y^i$ ,

$$\delta \Gamma^q = 0.$$

Hence, applying (2.11) to  $\Gamma^q$  we get

$$(5.8) \quad \Gamma^q = \delta \delta^{-1} \Gamma^q.$$

Next, we denote by  $\mu \in \Omega^2(\text{Mat}_N(\mathcal{S}M))[[\hbar]]$  the left-hand side of (5.6)

$$(5.9) \quad \mu = DB^q + \frac{1}{2}[B^q, B^q]_\diamond.$$

Applying  $\delta^{-1}$  to  $\mu$  we get

$$(5.10) \quad \delta^{-1} \mu = \delta^{-1} \left( \nabla B^q + A(B^q) + \frac{1}{2}[B^q, B^q]_\diamond \right) - \delta^{-1} \delta B^q.$$

On the other hand, due to (2.10),  $\delta^{-1} B^q = \delta^{-1} \Gamma^q$ . Hence, applying (2.11) to  $B^q$  and using (5.8) we get

$$\delta^{-1} \delta B^q = B^q - \Gamma^q.$$

Thus, in virtue of (5.5) and (5.10)

$$(5.11) \quad \delta^{-1} \mu = 0.$$

Using the equation  $D^2 = 0$  it is not hard to derive that

$$D\mu + [B^q, \mu]_\diamond = 0.$$

In other words

$$\delta \mu = \nabla \mu + A(\mu) + [B^q, \mu]_\diamond.$$

Therefore, applying (2.11) to  $\mu$  and using (5.11) we get

$$\mu = \delta^{-1} (\nabla \mu + A(\mu) + [B^q, \mu]_\diamond).$$

The latter equation has the unique vanishing solution due to (5.7). This argument concludes the proof of the proposition and gives us a flat connection of the form (5.4).  $\square$

The differential  $D^q$  (5.4) naturally extends to the DGLA  $\Omega^\bullet(C^\bullet(\text{Mat}_N(\mathcal{SM}[[\hbar]])))$  and to the DGLA module  $\Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{SM}[[\hbar]])))$ . Namely, on

$$\Omega^\bullet(C^\bullet(\text{Mat}_N(\mathcal{SM}[[\hbar]])))$$

the differential  $D^q$  is defined by the formula

$$(5.12) \quad D^q = D + [\partial_\diamond B^q, ]_G,$$

and on  $\Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{SM}[[\hbar]])))$  it is defined by the equation

$$(5.13) \quad D^q = D + R_{\partial_\diamond B^q}.$$

Here,  $\partial_\diamond$  is the Hochschild coboundary operator on  $C^\bullet(\text{Mat}_N(\mathcal{SM}[[\hbar]]))$  where  $\mathcal{SM}[[\hbar]]$  is considered with the product  $\diamond$  (4.4).

Let us prove an obvious analogue of [11], Lemma 1, p. 10:

**Lemma 2.** *If  $B^q$  is obtained by iterating Equation (5.5) then*

$$(5.14) \quad Dq + [B^q, q]_\diamond = 0.$$

*Proof.* Since  $q$  does not depend on fiber coordinates  $y^i$  Equation (5.14) boils down to

$$(5.15) \quad dq + [B^q, q] = 0,$$

where  $[ , ]$  stands for the ordinary matrix commutator.

On the other hand Equation (5.3) tells us that  $dq = -[\Gamma^q, q]$ . Thus it suffices to prove that

$$(5.16) \quad [B^q, q] - [\Gamma^q, q] = 0.$$

Let us denote the right-hand side of (5.16) by  $\Psi$

$$\Psi = [B^q, q] - [\Gamma^q, q].$$

Using Equations (5.3) and (5.6) it is not hard to show that

$$(5.17) \quad D\Psi + [B^q, \Psi] = 0.$$

On the other hand, Equations (2.10) and (5.5) imply that  $\delta^{-1}B^q = \delta^{-1}\Gamma^q$ , and hence,

$$\delta^{-1}\Psi = [\delta^{-1}B^q, q] - [\delta^{-1}\Gamma^q, q] = 0.$$

Therefore, applying (2.11) to  $\Psi$  and using (5.17) we get that

$$\Psi = \delta^{-1}(\nabla\Psi + A(\Psi) + [B^q, \Psi]_{\diamond}).$$

This equation has the unique vanishing solution due to (5.7).

The lemma is proved.  $\square$

We will need the following proposition:

**Proposition 10.** *There exists an element*

$$(5.18) \quad U \in \text{Mat}_N(\mathcal{SM}[[\hbar]])$$

such that<sup>7)</sup>

$$(5.19) \quad U = I \text{ mod } \text{Mat}_N(F^1 \mathcal{SM}[[\hbar]]),$$

and

$$(5.20) \quad D^q = D + [U^{-1} \diamond DU, ]_{\diamond},$$

where  $\diamond$  is the obvious extension of the product (4.4) to  $\text{Mat}_N(\mathcal{SM}[[\hbar]])$ .

*Proof.* To prove the proposition it suffices to construct an element  $U \in \text{Mat}_N(\mathcal{SM}[[\hbar]])$  satisfying the following equation:

$$U^{-1} \diamond DU = B^q,$$

or equivalently

$$(5.21) \quad DU - U \diamond B^q = 0.$$

We claim that a solution of (5.21) can be found by iterating the equation

$$(5.22) \quad U = 1 + \delta^{-1}(\nabla U + A(U) - U \diamond B^q).$$

Indeed, let us denote by  $\Phi$  the right-hand side of (5.21):

$$\Phi = DU - U \diamond B^q.$$

Due to (5.6)

$$(5.23) \quad D\Phi + \Phi \diamond B^q = 0.$$

On the other hand Equations (5.22), (2.10) and (2.11) for  $a = U$  imply that

$$(5.24) \quad \delta^{-1}\Phi = 0.$$

<sup>7)</sup> Equation (5.19) implies that  $U$  is invertible in the algebra  $\text{Mat}_N(\mathcal{SM}[[\hbar]])$  with the product  $\diamond$ .

Hence, applying identity (2.11) to  $a = \Phi$  and using (5.23) we get

$$\Phi = \delta^{-1}(\nabla\Phi + A(\Phi) + \Phi \diamond B^q).$$

Due to (5.7) the latter equation has the unique vanishing solution and the desired element  $U$  (5.18) is constructed.  $\square$

Due to Equation (5.14), Proposition 10 implies that the element

$$Q = U \diamond q \diamond U^{-1}$$

is flat with respect to the initial Fedosov differential (2.5). Therefore, by definition of the map  $\lambda_C$  (see [17], Eq. (5.1), chapter 5)

$$(5.25) \quad Q = \lambda_C(Q_0),$$

where

$$Q_0 = Q|_{y^i=0}.$$

Since  $Q$  is an idempotent in the algebra  $\text{Mat}_N(\mathcal{SM}[[\hbar]])$  with the product  $\diamond$  (4.4) the element  $Q_0$  is an idempotent of the algebra  $\text{Mat}_N(\mathcal{O}_M^{\hbar})$ .

Furthermore, due to (5.19)

$$Q_0|_{\hbar=0} = q$$

and hence, by Proposition 1,  $\text{ind}([Q_0]) = \text{ind}([P])$ .

By definition of the trace density map  $\text{trd}$  (4.12) the class  $\text{ind}([Q_0])$  is represented by the cycle

$$(5.26) \quad c = \tilde{\mathcal{S}}_0(\lambda_C(\text{tr } Q_0)),$$

of the complex  $\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]]$  with the differential (4.2). Here  $\tilde{\mathcal{S}}_0$  is the structure map of the zeroth level of the quasi-isomorphism  $\tilde{\mathcal{S}}$  in (4.9).

Due to (5.25)

$$(5.27) \quad c = \tilde{\mathcal{S}}_0(\text{tr } Q),$$

where  $Q = U \diamond q \diamond U^{-1}$ .

Let us prove that

**Proposition 11.** *The cycles*

$$(5.28) \quad Q = U \diamond q \diamond U^{-1}$$

and

$$(5.29) \quad \tilde{Q} = \sum_{k \geq 0} (-1)^k [q \otimes (B^q)^{\otimes(2k)} + q \otimes (B^q)^{\otimes(2k+1)}]$$

are homologous in the complex  $\Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]])))$  with the differential  $D + \mathfrak{b}_\diamond$ .

*Proof.* A direct computation shows that

$$Q - \tilde{Q} = D\psi + \mathfrak{b}_\diamond\psi$$

where

$$\psi = \sum_{k \geq 0} (-1)^k [U \otimes (B^q)^{\otimes(2k)} \otimes q \diamond U^{-1} + U \otimes (B^q)^{\otimes(2k+1)} \otimes q \diamond U^{-1}]. \quad \square$$

It is not hard to show that

$$\tilde{Q} = \exp(R_{B^q})(q).$$

Therefore, the class  $\text{ind}([P])$  is represented by the cycle

$$(5.30) \quad c' = \tilde{\mathcal{S}}_0 \circ \text{tr} \circ \exp(R_{B^q})(q).$$

Let us consider the following diagram of quasi-isomorphisms of DGLA modules:

$$(5.31) \quad \begin{array}{ccccc} \Omega^\bullet(\mathcal{F}_{\text{poly}}^\bullet[[\hbar]]) & \xrightarrow{\tilde{\mathcal{X}}} & \Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]] & \xrightarrow{\text{cotr}'} & \Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]]))) \\ \downarrow L_{\text{mod}} & & \downarrow R_{\text{mod}} & & \downarrow R_{\text{mod}} \\ \Omega^\bullet(\mathcal{E}^\bullet[[\hbar]]) & \xleftarrow{\tilde{\mathcal{Y}}} & \Omega^\bullet(C_\bullet(\mathcal{S}M))[[\hbar]] & \xleftarrow{\text{tr}'} & \Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]]))) \end{array}$$

where  $\Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]])))$  and  $\Omega^\bullet(C_\bullet(\text{Mat}_N(\mathcal{S}M[[\hbar]])))$  carry respectively the differentials (5.12) and (5.13), the rest DGLAs and DGLA modules carry the same differentials as in (4.9), and

$$(5.32) \quad \text{cotr}' = \exp(-[B^q, \ ]_G) \circ \text{cotr}, \quad \text{tr}' = \text{tr} \circ \exp(R_{B^q}).$$

Recall that  $E$  is the vector bundle corresponding to the idempotent  $q$  of the algebra  $\text{Mat}_N(\mathcal{O}_M)$ . In other words the rank  $N$  trivial bundle is the direct sum

$$(5.33) \quad I_N = E \oplus \bar{E},$$

where  $\bar{E}$  is the bundle corresponding to  $1 - q$ .

In a trivialization compatible with the decomposition (5.33), the endomorphism  $q$  is represented by the constant matrix

$$(5.34) \quad \tilde{q} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix},$$

where  $m$  is the rank of the bundle  $E$ .

If we choose different trivializations on  $I_N$  the element  $B^q$  in (5.4) may no longer be regarded as a one-form with values in  $\mathcal{S}M \otimes \text{End}(I_N)[[\hbar]]$ . Instead  $B^q$  is the sum

$$B^q = \Gamma^q + \tilde{B}^q$$

of the connection form  $\Gamma^q$  (5.2) and an element  $\tilde{B}^q \in \Omega^1(\mathcal{S}M \otimes \text{End}(I_N))[[\hbar]]$  such that

$$\tilde{B}^q|_{y=0} = 0.$$

However the maps  $\text{cotr}'$  and  $\text{tr}'$  (5.32) in Diagram (5.31) are still well defined. The latter follows from the fact that we deal with normalized Hochschild (co)chains.

Lemma 2 implies that, in a trivialization compatible with the decomposition (5.33), the form  $B^q$  is represented by the block diagonal matrix

$$(5.35) \quad B^q = \begin{pmatrix} A^E & 0 \\ 0 & A^{\bar{E}} \end{pmatrix},$$

where  $A^E$

$$\begin{aligned} A^E &= \Gamma^E + \tilde{A}^E, \\ A^{\bar{E}} &= \Gamma^{\bar{E}} + \tilde{A}^{\bar{E}}, \end{aligned}$$

$\Gamma^E$  (resp.  $\Gamma^{\bar{E}}$ ) is a connection form of  $E$  (resp.  $\bar{E}$ ) and  $\tilde{A}^E \in \Omega^1(\mathcal{S}M \otimes \text{End}(E))[[\hbar]]$ ,  $\tilde{A}^{\bar{E}} \in \Omega^1(\mathcal{S}M \otimes \text{End}(\bar{E}))[[\hbar]]$ .

The latter implies that the cycle  $c'$  (5.30)

$$c' = \tilde{\mathcal{S}}_0 \circ \text{tr} \circ \exp(R_{A^E})(1_E).$$

Hence  $c'$  represents the class  $H_\bullet(\lambda_{\mathcal{A}})(\text{trd}_E([1_E]))$  in the cohomology of the complex  $\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]]$  with the differential  $D + L_{\lambda_T(\pi)}$ . Here  $\lambda_{\mathcal{A}}$  is the embedding of  $\mathcal{A}^\bullet[[\hbar]]$  into  $\Omega^\bullet(\mathcal{E}^\bullet)[[\hbar]]$  (2.13) and  $H_\bullet$  denotes the cohomology functor.

Since  $c'$  is cohomologous to  $c$  (5.26) the statement of Theorem 1 follows.  $\square$

**Remark.** Theorem 1 can be easily generalized to the deformation quantization of the algebra  $\mathcal{O}_M^{\mathbb{C}}$  of smooth complex valued functions. In this setting we should use the corresponding analogue of the formality theorem from [16] for smooth complex vector bundles.

## 6. Concluding remarks

**6.1. The relation to the cyclic version of the algebraic index theorem.** In [43] D. Tamarkin and B. Tsygan, inspired by the Connes-Moscovici higher index formulas [14], suggested the first version of the algebraic index theorem for a Poisson manifold. This version is based on the cyclic formality conjecture [44].

The statement of this conjecture [44] (see Conjecture 3.3.2) would provide us with the cyclic version of the trace density map which is a map

$$(6.1) \quad \text{trd}_{\bullet}^{\text{cyc}} : HC_{\bullet}^{\text{per}}(\mathcal{O}_M^{\hbar}) \rightarrow H_{\bullet}(\Omega^{\bullet}(M)((u))[[\hbar]], u d)$$

from the periodic cyclic homology  $HC_{\bullet}^{\text{per}}(\mathcal{O}_M^{\hbar})$  of the deformation quantization algebra  $\mathcal{O}_M^{\hbar}$  to the homology of the complex

$$(6.2) \quad (\Omega^{\bullet}(M)((u))[[\hbar]], u d)$$

where  $u$  is an auxiliary variable of degree  $-2$  and  $d$  is the De Rham differential.

The algebraic index theorem [43] of D. Tamarkin and B. Tsygan describes the map (6.1) in terms of the principal symbol map

$$(6.3) \quad \sigma_{\text{cyc}} : HC_{\bullet}^{\text{per}}(\mathcal{O}_M^{\hbar}) \rightarrow HC_{\bullet}^{\text{per}}(\mathcal{O}_M)$$

and characteristic classes of  $M$ .

In order to show how our quantum index density (1.5) fits into the picture of D. Tamarkin and B. Tsygan let us recall that the map (6.1) is the composition of two isomorphisms:

$$\text{trd}_{\bullet}^{\text{cyc}} = \beta \circ \widetilde{\text{trd}}_{\bullet}^{\text{cyc}}.$$

The first isomorphism is the map<sup>8)</sup>

$$(6.4) \quad \widetilde{\text{trd}}_{\bullet}^{\text{cyc}} : HC_{\bullet}^{\text{per}}(\mathcal{O}_M^{\hbar}) \rightarrow H_{\bullet}(\Omega^{\bullet}(M)((u))[[\hbar]], L_{\pi} + u d)$$

from the periodic cyclic homology of  $\mathcal{O}_M^{\hbar}$  to the homology of the complex

$$(6.5) \quad (\Omega^{\bullet}(M)((u))[[\hbar]], L_{\pi} + u d),$$

where  $L_{\pi}$  denotes the Lie derivative along the bivector  $\pi$  (1.1).

The second isomorphism

$$(6.6) \quad \beta : H_{\bullet}(\Omega^{\bullet}(M)((u))[[\hbar]], L_{\pi} + u d) \rightarrow H_{\bullet}(\Omega^{\bullet}(M)((u))[[\hbar]], u d)$$

is induced by the map between complexes (6.2) and (6.5)

$$c \rightarrow \exp(u^{-1}i_{\pi})c,$$

where  $i_{\pi}$  denotes the contraction with the bivector  $\pi$ .

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<sup>8)</sup> It is the cyclic formality conjecture which would imply the existence of the isomorphism (6.4).



The quantum index density (1.5) fits into the following commutative diagram:

$$\begin{array}{ccc}
 K_0(\mathcal{O}_M^\hbar) & \xrightarrow{\text{ch}_0^-} & HC_0^-(\mathcal{O}_M^\hbar) = HC_0^{\text{per}}(\mathcal{O}_M^\hbar) \\
 \downarrow \text{ind} & & \downarrow \widetilde{\text{tr}}_0^{\text{cyc}} \\
 HP_0(M, \pi) & \xleftarrow{u=0} & H_0(\Omega^\bullet(M)[[u]][[\hbar]], L_\pi + u d) \\
 (6.7) & & \downarrow = \\
 & & H_0(\Omega^\bullet(M)((u)[[\hbar]], L_\pi + u d) \\
 & & \downarrow \beta \\
 & & H_0(\Omega^\bullet(M)((u)[[\hbar]], u d),
 \end{array}$$

where  $\text{ch}_0^-$  denotes the Chern character map (see [30], Proposition 8.3.8, Section 8.3).

We would like to mention recent paper [10] by A. S. Cattaneo and G. Felder. In this paper the authors consider a manifold  $M$  equipped with a volume form and construct an  $L_\infty$  morphism from the DG Lie algebra module  $CC_\bullet^-(\mathcal{O}_M)$  of negative cyclic chains of  $\mathcal{O}_M$  to a DG Lie algebra module modeled on polyvector fields using the volume form. Although this  $L_\infty$  morphism is not a quasi-isomorphism one can still use it to construct a specific trace on the deformation quantization algebra of a unimodular Poisson manifold. It would be interesting to find a formula for the index map corresponding to this trace.

**6.2. Lie algebroids and the algebraic index theorem.** There are two ways when the Lie algebroid theory comes in the game. The first one, more direct, is based on the formality theorem for Lie algebroids (see [8] and [9] applied to deformation quantization of the so-called Poisson-Lie algebroid  $F \mapsto M$  with a bracket  $[\cdot, \cdot]$  and anchor  $a : \Gamma(M, F) \mapsto \Gamma(M, TM)$ . Such algebroid carries on its fibers a ‘‘Poisson bivector’’  $\pi^F \in \Gamma(M, \Lambda^2(F))$  satisfying the Jacobi identity:  $[\pi^F, \pi^F] = 0$ . The corresponding version of the algebraic index theorem in this setting could be considered as a generalization of the results of R. Nest and B. Tsygan from [34].

The second way concerns the natural Poisson bracket on the *dual* vector bundle  $F^* \mapsto M$  of a Lie algebroid  $F \mapsto M$ .

More concretely, we will be interested in the case when the Lie algebroid comes as the Lie algebroid  $F(\mathcal{G}) \mapsto \mathcal{G}_0$  associated to a Lie groupoid  $\mathcal{G} \rightrightarrows \mathcal{G}_0$ . The dual bundle  $F^*(\mathcal{G})$  carries a natural Poisson structure which is a direct generalization of the canonical symplectic structure on  $T^*M$  and Lie-Poisson structure on the dual space  $\mathfrak{g}^*$  of a Lie algebra  $\mathfrak{g}$ . (We should remark that the simplest Lie algebroid  $TM \mapsto TM$ ,  $a = \text{id}$  is associated with the pair groupoid  $\mathcal{G} = M \times M$ .)

Following standard definitions of [35] we will associate to a Lie algebroid  $F \mapsto M$  the *adiabatic* Lie algebroid  $F_\hbar \mapsto M \times I$  whose total space is the pull-back of  $F$  and the bracket  $[\cdot, \cdot]_\hbar := \hbar[\cdot, \cdot]$ . It is interesting and important result that this Lie algebroid comes as a Lie algebroid of Connes’s tangent groupoid  $\mathcal{G}^{\mathcal{F}}$  (see [12]).

There are two  $C^*$ -algebras that can be considered in this situation. The first one is Connes’s  $C^*$ -algebra  $C^*(\mathcal{G})$  of a Lie groupoid and the second one is its “classical” counterpart—the Poisson algebra  $C_0(F^*(\mathcal{G}))$  of continuous functions on  $F^*(\mathcal{G})$ . An appropriate type of a deformation quantization in the  $C^*$ -algebra context was proposed by M. Rieffel [37]. Let us remind it here omitting some technical details.

**Definition 1.** A  $C^*$ -algebraic deformation quantization (or the “strict” Rieffel’s quantization) of a Poisson manifold  $M$  is a continuous family of  $C^*$ -algebras  $(A, A_{\hbar}, \hbar \in I = [0, 1])$  such that  $A_0 = C_0(M)$  and the Poisson algebra  $\tilde{A}_0$  is dense in  $C_0(M)$ . There is a family of sections

$$\mathfrak{F} : I \mapsto \bigsqcup_{\hbar \in I} A_{\hbar}, \quad \{\mathfrak{F}(\hbar) \mid \mathfrak{F} \in A\} = A_{\hbar}$$

and the function  $\hbar \rightarrow \|\mathfrak{F}(\hbar)\|$  continuous. Each algebra  $A_{\hbar}$  is equipped with  $*_{\hbar}$ -product, a norm  $\|\cdot\|_{\hbar}$  and  $*_{\hbar}$ -involution. The map

$$q_{\hbar}(f) = f : \tilde{A}_0 \mapsto A_{\hbar}$$

satisfies the following axiom (the “correspondence principle”):

$$\lim_{\hbar \rightarrow 0} \left\| \frac{i}{\hbar} [q_{\hbar}(f), q_{\hbar}(g)]_{\hbar} - q_{\hbar}(\{f, g\}) \right\|_{\hbar} = 0.$$

The proper  $C^*$  analog of the algebra  $A[[\hbar]]$  is a  $C[I]C^*$ -algebra (see [27]) and we will identify  $A_{\hbar}$  with  $A/C[I, \hbar]A$  where  $C[I, \hbar] := \{\mathfrak{F} \in C[I] \mid \mathfrak{F}(\hbar) = 0\}$ .

We will denote by  $\text{pr}_{\hbar} : A \mapsto A_{\hbar}$  the canonical projection and we will not distinguish between  $a \in A$  and the section  $a : \hbar \rightarrow \text{pr}_{\hbar}(a)$ . Now there is a section map  $q : \tilde{A}_0 \mapsto A$  such that  $q_{\hbar} = \text{pr}_{\hbar} \circ q$ .

In concrete situation which we are interested in, the  $C^*$ -deformation quantization was studied by N. Landsman [28]. Taking  $\hbar \in I = [0, 1]$ , for any Lie groupoid  $\mathcal{G}$  the field

$$A_0 := C_0(F^*(\mathcal{G})), \quad A_{\hbar} = C^*(\mathcal{G}), \quad A = C^*(\mathcal{G}^{\mathcal{T}})$$

(where  $\mathcal{G}^{\mathcal{T}}$  is the tangent groupoid of  $\mathcal{G}$ ), we obtain a  $C^*$  algebraic deformation quantization of the Lie algebroid  $F^*(\mathcal{G})$ .

In this context, the arrow of “symbol map”  $\sigma$  in Diagram (1.11) (which in fact provides an isomorphism of the  $K$ -groups) admits an “inversion”:

$$(6.8) \quad \text{ind}_a := \sigma^{-1} : K_0(F^*(\mathcal{G})) \rightarrow K_0(C^*(\mathcal{G})),$$

which is called the *analytic index map* (see [40], [31] and [32]). This map plays a key role in Connes’s generalization of the Atiyah-Singer index theorem in the non-commutative geometry. Proposition 1 (or more generally, Rosenberg’s theorem [39]) gives us an isomorphism of  $K$ -groups  $K_0(\tilde{A}[[\hbar]]) \cong K_0(\tilde{A})$ .

In this setting we propose a plausible

**Conjecture 1.** *The maps  $\text{ind}$  and  $\text{ind}_c$  from Diagram (5.1) in Theorem 1 are well defined in the setting of the strict quantization. Furthermore, the diagram*

$$(6.9) \quad \begin{array}{ccc} K_0(C^*(\mathcal{G})) & \xrightarrow{\text{ind}} & HP_0(F^*(\mathcal{G}), \pi) \\ & \swarrow \text{ind}_a & \searrow \text{ind}_c \\ & K_0(F^*(\mathcal{G})) & \end{array}$$

is commutative.

Let us discuss this statement in two important cases (see [12] and [28]):

(1) When  $\mathcal{G}$  is Connes’s tangent groupoid

$$\mathcal{G}^{\mathcal{F}} = \mathcal{G}_1 \sqcup \mathcal{G}_2 = (M \times M \times (0, 1]) \sqcup TM,$$

the corresponding Lie algebroid  $F = TM$  and  $F^* = T^*M$ . We suppose that the manifold has a Riemannian metric and denote by  $\mathcal{K}(L^2(M))$  the algebra of compact operators on the Hilbert space  $L_2(M)$  of square-integrable functions on  $M$ .

The strict quantization in this case coincides with the Moyal deformation.

The associated  $C^*$ -algebras in this case are:  $A_0 = C^*(TM)$  and  $A_{\hbar} = C^*(M \times M)$ ,  $\forall \hbar \in (0, 1]$ . The first algebra is identified (via the Fourier transform) with  $C_0(T^*M)$  and the second one is identified with the algebra  $\mathcal{K}(L^2(M))$ . Due to [12], II.5, Prop. 5.1, we have the exact sequence of  $C^*$ -algebras:

$$0 \rightarrow C^*(\mathcal{G}_1) \rightarrow C^*(\mathcal{G}^{\mathcal{F}}) \xrightarrow{\tilde{\sigma}} C^*(\mathcal{G}_2) \rightarrow 0$$

or in other terms

$$0 \rightarrow C_0((0, 1]) \otimes \mathcal{K}(L^2(M)) \rightarrow C^*(\mathcal{G}^{\mathcal{F}}) \xrightarrow{\tilde{\sigma}} C^*(C_0(T^*M)) \rightarrow 0$$

and from the long exact sequence in  $K$ -theory we can obtain the map

$$\tilde{\sigma}_* : K_0(\mathcal{G}^{\mathcal{F}}) \simeq K_0(C_0(T^*M)) = K^0(T^*M).$$

The map  $\text{ind}_a$  is nothing but the Atiyah-Singer analytic index

$$\text{ind}_a = \text{tr} \circ \iota_* \circ (\tilde{\sigma}_*)^{-1} : K^0(T^*M) \rightarrow \mathbb{Z}$$

with

$$\iota : M \times M \rightarrow \mathcal{G}^{\mathcal{F}}, \quad \iota(x, y) = (x, y, 1), \quad x, y \in M$$

and

$$C^*(\mathcal{G}^{\mathcal{F}}) \xrightarrow{\iota_*} C^*(M \times M) = \mathcal{K}(L^2(M)) \stackrel{\text{tr}}{\cong} \mathbb{Z}.$$

Conjecture 1 transforms in the following commutative diagram:

$$(6.10) \quad \begin{array}{ccc} & \mathbb{Z} & \\ \text{tr} \nearrow & & \searrow \int \\ K_0(\mathcal{K}(L^2(M))) & \xrightarrow{\text{ind}} & H_c^{2n}(T^*M) \\ \nwarrow \iota_* \circ (\sigma_*)^{-1} & & \nearrow \text{ind}_c \\ & K^0(T^*M) & \end{array}$$

Here we use the fact that the canonical Poisson structure  $\pi$  on  $T^*M$  is symplectic and hence

$$HP_0(C_0(T^*M), \pi) \simeq H_c^{2n}(T^*M)$$

where the De Rham cohomology with compact support is used and the map from  $H_c^{2n}(T^*M)$  to  $\mathbb{Z}$  is given by the usual integral of top degree forms over  $M$ .

(2) If  $\mathcal{G}_0$  is a point and  $\mathcal{G} = G$  is a Lie group then the associated Lie algebroid is nothing but the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$  and the dual  $F^* = \mathfrak{g}^*$ . In this case the associated  $C^*$ -algebras are  $A_0 = C^*(\mathfrak{g}) \simeq C_0(\mathfrak{g}^*)$  (again, via the Fourier transform) and  $A_{\hbar} = C^*(G)$ ,  $\forall \hbar \in (0, 1]$  is the usual convolution algebra of  $G$  defined by a Haar measure.

The map  $\sigma^{-1}$  is the composition

$$K_0(C_0(\mathfrak{g}^*)) \xrightarrow{F^*} K_0(C^*(\mathfrak{g})) \xrightarrow{\text{exp}^*} K_0(C^*(G)),$$

where  $\text{exp} : \mathfrak{g} \mapsto G$  is the usual exponential map and the strict deformation quantization of the Poisson-Lie structure in  $\mathfrak{g}^*$  was proposed by Rieffel in [38].

Diagram (6.9) takes the following form:

$$(6.11) \quad \begin{array}{ccc} K_0(C^*(G)) & \xrightarrow{\text{ind}} & HP_0(\mathfrak{g}^*, \pi) \\ \nwarrow \sigma^{-1} & & \nearrow \text{ind}_c \\ & K_0(C_0(\mathfrak{g}^*)) & \end{array}$$

We would like to stress that our index theorem (1.11) and the conjectural “3-ind”-theorem (6.9) have, in fact, the same flavor of “index-without-index” theorems like the index theorems in the theory related to Baum-Connes conjecture. A deformation aspect of the Baum-Connes is discussed in [28].

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