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Equivariant Cohomology and Localization for Lie Algebroids[*]

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ABSTRACT. Let M be a manifold carrying the action of a Lie group G, and let A be a Lie algebroid on M equipped with a compatible infinitesimal G-action. Using these data, we construct an equivariant cohomology of A and prove a related localization formula for the case of compact G. By way of application, we prove an analog of the Bott formula.

KEY WORDS: Lie algebroid, equivariant cohomology, localization formula.

1. Introduction

The notion of Lie algebroid, which may be regarded as a natural generalization of the tangent bundle to a manifold, allows one to treat several geometric structures, such as Poisson manifolds, connections on principal bundles, and foliations, in a unified manner.

Since in many situations one can associate a Lie algebroid with a singular foliation, Lie algebroids seem to provide a way for generalizing several constructions (e.g., connections) to singular settings. Recently, several field-theoretic models based on Lie algebroids have been proposed; e.g., see [23].

Every Lie algebroid intrinsically defines a cohomology theory. On the other hand, the theory of G-differential complexes developed in [12] encompasses the case of *equivariant Lie algebroid cohomology*, which can be defined if a Lie algebroid carries a (possibly infinitesimal) action of a Lie group G compatible with an action of G on the base manifold M.

A natural question arises as to whether one can generalize the usual localization formula for equivariant (de Rham) cohomology to this setting. This also seems to be of certain interest for applications; for instance, recently the localization formula has been used to compute the partition function of $N = 2$ super Yang–Mills theory ([19], [4]). In this case, the relevant cohomology is the equivariant de Rham cohomology of the instanton moduli space—the moduli space of framed self-dual connections on $\mathbb{R}^4$. (That is, the relevant Lie algebroid is the tangent bundle to the instanton moduli space.) It seems plausible that super Yang–Mills theories with various numbers of supersymmetry charges can be treated in a unified way for various choices of a Lie algebroid structure on the instanton moduli space.

In this paper, we present a localization formula for the equivariant cohomology of a Lie algebroid for the case in which this Lie algebroid A on a (compact oriented) manifold M carries an (infinitesimal) action of a Lie group. If one twists such cohomology by the orientation bundle naturally associated with A, then equivariant cocycles can be integrated over M. If the group action has only isolated fixed points, then the value of the integral can be calculated as a finite sum of suitably defined residues at the fixed points.

This localization formula is of course reduced to the usual one for equivariant de Rham cohomology when the Lie algebroid A is the tangent bundle TM. In a similar way, it encompasses a number of classical localization formulas, providing new proofs for them. For instance, it implies a generalization to the Lie algebroid setting of the Bott theorem about zeros of vector fields (and also related formulas due to Cenkl and Kubarski [9], [15]). Our formula generalizes the Bott formula

exactly in the same sense as the notion of Lie algebroid generalizes that of tangent bundle. Moreover, if M is complex and A is the Atiyah algebroid of a holomorphic vector bundle E on M, our localization formula can be specialized so as to produce localization formulas due to Baum–Bott [1], Chern [10], Carrell–Lieberman [8], [7], and Liu [16]. These aspects will be developed in [5]. Another localization formula for the Atiyah algebroid, which can be obtained from our formula as well, is given in [22]. Many further examples and applications can be given.

The paper is organized as follows. In Section 2, we present the basic definitions and some constructions concerning Lie algebroid cohomology. Section 3 introduces equivariant Lie algebroid cohomology, basically within the framework of the theory of *G-differential complexes* developed in [12]. Moreover, we prove our localization formula. In Section 4, following [17], we consider one possible approach to the definition of connections on Lie algebroids and use this to construct equivariant characteristic classes for Lie algebroids. This will be used in Section 5 to prove an analog of the Bott formula.

2. Lie Algebroid Cohomology

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ the space of vector fields on M equipped with the usual Lie bracket $[\,\cdot\,,\,\cdot\,]$.

Definition 2.1. An algebroid A on M is a vector bundle on M together with a vector bundle morphism $a\colon A \to TM$ (called the anchor) and a structure of Lie algebra on the space of global sections $\Gamma(A)$ such that

(i) $a\colon \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

(ii) The Leibniz rule

$$\{\alpha, f\beta\} = f\{\alpha, \beta\} + a(\alpha)(f)\beta$$

holds for any α, $\beta \in \Gamma(A)$ and any function f. (Here $\{\,\cdot\,,\,\cdot\,\}$ is the bracket in $\Gamma(A)$.)

Morphisms between two Lie algebroids (A, a) and (A', a') on a same base manifold M are defined in a natural way; i.e., they are vector bundle morphisms $\phi\colon A \to A'$ such that the map $\phi\colon \Gamma(A) \to \Gamma(A')$ is a Lie algebra homomorphism and the obvious diagram involving the two anchors commutes.

With any Lie algebroid A one can associate the cohomology complex $(C_A^\bullet, \delta)$ with $C_A^\bullet = \Gamma(\Lambda^\bullet A^*)$ and differential δ defined by

$$(\delta\xi)(\alpha_1, \ldots, \alpha_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1}a(\alpha_i)(\xi(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1}))$$
$$+ \sum_{i<j}(-1)^{i+j}\xi(\{\alpha_i, \alpha_j\}, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{p+1}),$$

where $\xi \in C_A^p$ and $\alpha_i \in \Gamma(A)$, $1 \leqslant i \leqslant p+1$. The resulting cohomology is denoted by $H^\bullet(A)$ and is called the cohomology of the Lie algebroid A.

Remark 2.2. We point out that if A is a vector bundle, δ is a derivation of degree $+1$ of the graded algebra $\Gamma(\Lambda^\bullet A^*)$, and $\delta^2 = 0$, then, using δ, one can construct an anchor $a\colon A \to TM$ and a Lie bracket on $\Gamma(A)$ making A a Lie algebroid. One simply defines

$$a(\alpha)(f) = \delta f(\alpha) \quad \text{for } \alpha \in \Gamma(A), f \in C^\infty(M),$$

$$\xi(\{\alpha, \beta\}) = a(\alpha)(\xi(\beta)) - a(\beta)(\xi(\alpha)) - \delta\xi(\alpha, \beta) \quad \text{for } \alpha, \beta \in \Gamma(A), \xi \in \Gamma(A^*). \qquad \square$$

Recall some examples of Lie algebroids.

Example 2.3. An involutive distribution on a tangent bundle (i.e., a foliation) is a Lie algebroid with injective anchor. $\qquad\square$

Example 2.4 [24]**.** Let $\mathfrak{M} = (M, \mathscr{F})$ be a supermanifold; in particular, M is a differentiable manifold and $\mathscr{F}$ is a sheaf of $\mathbb{Z}_2$-graded commutative $\mathbb{R}$-algebras on M that can be realized as the sheaf of sections of the exterior algebra bundle $\wedge^\bullet E$ for a vector bundle E. Let D be an odd supervector field on $\mathfrak{M}$ with $D^2 = 0$. Then E^* with the anchor and the Lie algebra structure on $\Gamma(E^*)$ given, according to Remark 2.2, by D regarded as a differential for the complex $\Gamma(\wedge^\bullet E)$ is a Lie algebroid. Of course, starting from a Lie algebroid we can construct a supermanifold with an odd supervector field on it squaring to zero, so that the two sets of data are equivalent.

Example 2.5. Let (M, Π) be a Poisson manifold, where Π is a Poisson tensor. In this case, $A = T^*M$ with the Lie bracket

$$\{\alpha, \beta\} = \mathscr{L}_{\Pi(\alpha)}\beta - \mathscr{L}_{\Pi(\beta)}(\alpha) - d\Pi(\alpha, \beta) \tag{1}$$

of differential forms (where $\mathscr{L}$ is the Lie derivative), and the anchor is the Poisson tensor (more precisely, the corresponding Hamiltonian map). The cohomology of A is the Lichnerowicz–Poisson cohomology of (M, Π). $\qquad\square$

Example 2.6. Let $P \xrightarrow{p} M$ be a principal bundle with structure group G. One has the Atiyah exact sequence

$$0 \to \mathrm{ad}(P) \to TP/G \to TM \to 0 \tag{2}$$

of vector bundles on M. (A connection on P is a splitting of this sequence.) Sections of the vector bundle TP/G are in a one-to-one correspondence with G-invariant vector fields on P. On the global sections of TP/G, there is a natural Lie algebra structure, and taking the projection $TP/G \to TM$ for the anchor map, we obtain a Lie algebroid, the *Atiyah algebroid* associated with the principal bundle P.

With a vector bundle E on M, we can also associate an Atiyah algebroid. Indeed, in this case one has the short exact sequence

$$0 \to \mathrm{End}(E) \to \mathrm{Diff}^1_0(E) \to TM \to 0, \tag{3}$$

where $\mathrm{Diff}^1_0(E)$ is the bundle of first-order differential operators on E with scalar symbol [14], and again $\mathrm{Diff}^1_0(E)$, with the natural Lie algebra structure on its global sections and the natural map $\mathrm{Diff}^1_0(E) \to TM$ as the anchor, is a Lie algebroid. The two notions of Atiyah algebroid coincide if P is the bundle of linear frames of a vector bundle E. (Indeed, an element in $T_u P$ is given by an endomorphism of the fiber $E_{p(u)}$ and a vector in $T_{p(u)}M$.)

Lie algebroids whose anchor map is surjective, as in the case of Atiyah algebroids, are said to be *transitive*. $\qquad\square$

Following [11], we now describe a twisted form of the Lie algebroid cohomology together with a natural pairing between the two cohomologies. This will be another ingredient of the localization formula. Let Q_A be the line bundle $\wedge^r A \otimes \Omega^m_M$, where $r = \mathrm{rk}\, A$ and $m = \dim M$, and let Ω^m_M be the bundle of differential m-forms on M. (Later on, by $\Omega^m(M)$ we denote the space of global sections of this bundle.) For every $s \in \Gamma(A)$, define a map $L_s = \{s, \cdot\} \colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^\bullet A^*)$ by setting

$$L_s(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^k s_1 \wedge \cdots \wedge \{s, s_i\} \wedge \cdots \wedge s_k.$$

Moreover, we can define the map

$$D \colon \Gamma(Q_A) \to \Gamma(A^* \otimes Q_A) = \Gamma(A^*) \otimes_{C^\infty(M)} \Gamma(Q_A), \qquad D\tau(s) = L_s(X) \otimes \mu + X \otimes \mathscr{L}_{a(s)}\mu,$$

where $\tau = X \otimes \mu \in \Gamma(Q_A)$ and $s \in \Gamma(A)$. Consider the twisted complex $\widetilde{C}^\bullet_A = \Gamma(\wedge^\bullet A^* \otimes Q_A) = C^\bullet_A \otimes_{C^\infty(M)} \Gamma(Q_A)$, $C^\bullet_A = \bigoplus_{k=0}^r \Gamma(\wedge^k A^*)$, with differential $\tilde{\delta}$ defined by

$$\tilde{\delta}(\xi \otimes \tau) = \delta\xi \otimes \tau + (-1)^{\deg(\xi)}\xi \otimes D\tau, \qquad \xi \in C^\bullet_A.$$

We denote the resulting cohomology by $H^\bullet(A, Q_A)$.

There is a naturally defined map* $p\colon \widetilde{C}^\bullet_A \to \Omega^{\bullet-r+m}(M)$,

$$p(\psi \otimes X \otimes \mu) = (a(\psi \lrcorner X))\lrcorner\mu.$$

Proposition 2.7. *The morphism p is a chain map*** *in the sense that the diagram*

$$
\begin{array}{ccc}
\widetilde{C}^k_A & \xrightarrow{\quad p \quad} & \Omega^{k-r+m}(M) \\
\Big\downarrow{\tilde\delta} & & d\Big\downarrow \\
\widetilde{C}^{k+1}_A & \xrightarrow{\quad p \quad} & \Omega^{k-r+m+1}(M)
\end{array}
\tag{4}
$$

commutes up to sign; i.e., on C^k_A one has

$$p \circ \tilde\delta = (-1)^k d \circ p. \tag{5}$$

Proof. The Lie derivative on $\widetilde{C}^\bullet_A$ defined by $L_s = i_s \circ \tilde\delta + \tilde\delta \circ i_s$ for $s \in \Gamma(A)$ satisfies the commutation relation

$$p \circ L_s = \mathscr{L}_{a(s)} \circ p \tag{6}$$

on $\widetilde{C}^k_A$. By using this identity, we can prove (5) by descending induction on k. For $k = r$, the identity is reduced to $0 = 0$. For smaller k, it suffices to prove the identity for those $c' \in \widetilde{C}^k_A$ which can be represented as $c' = i_s c$ for some $s \in \Gamma(A)$ and $c \in \widetilde{C}^{k+1}_A$. In this case, the result follows from a simple computation which uses (6).

Let M be compact and oriented. Note that since $\widetilde{C}^r_A \simeq \Omega^m(M)$ (canonically), one can integrate elements of $\widetilde{C}^r_A$ over M. There is a nondegenerate pairing

$$C^k_A \otimes_{C^\infty(M)} \widetilde{C}^{r-k}_A \to \mathbb{R}, \qquad \xi \otimes (\psi \otimes X \otimes \mu) \mapsto \int_M (\xi \wedge \psi, X)\mu.$$

A version of Stokes' theorem holds for the complex $\widetilde{C}^\bullet_A$ [11]: if $c \in \widetilde{C}^{r-1}_A$, then

$$\int_M \tilde\delta c = 0.$$

This formula follows from identity (5) for $k = r - 1$. In turn, it implies that the pairing descends to cohomology, yielding a bilinear map

$$H^\bullet(A) \otimes H^{r-\bullet}(A, Q_A) \to \mathbb{R}. \tag{7}$$

This pairing in general may be degenerate.

One also has a natural morphism $C^\bullet_A \otimes_{C^\infty(M)} \widetilde{C}^\bullet_A \to \widetilde{C}^\bullet_A$ compatible with the degrees. Again, this descends to cohomology and provides a cup product

$$H^i(A) \otimes H^j(A, Q_A) \to H^{i+j}(A, Q_A). \tag{8}$$

3. Equivariant Cohomology and Localization

In this section, we introduce equivariant cohomology for Lie algebroids, basically following the pattern exploited in [12] to define equivariant cohomology for Poisson manifolds (and falling within the general theory of equivariant cohomology for G-differential complexes developed there). Moreover, we write out a localization formula for the equivariant Lie algebroid cohomology (Theorem 3.2).

We assume that there is an action of a Lie algebra $\mathfrak{g}$ on the Lie algebroid A, i.e., that there is a Lie algebra map

$$b\colon \mathfrak{g} \to \Gamma(A). \tag{9}$$

---

*We interchangeably use the notation $\psi\lrcorner$ and i_ψ for the inner product by an element ψ, according to what we feel aesthetically admissible.

**We are thankful to A. Rosly for pointing out this fact and for suggesting the following proof.

By composing this with the anchor map, we obtain a Lie algebra homomorphism $\tilde{\rho} = a \circ b \colon \mathfrak{g} \to \mathfrak{X}(M)$, i.e., an action of $\mathfrak{g}$ on M. Lie algebra maps like our b were introduced in [18] for the case of Atiyah algebroids under the name of "derivation representations."

Example 3.1 (cf. [25]). Let Π be a regular Poisson tensor on M (i.e., Π has constant rank), and let $\mathscr{S} = \mathrm{Im}(\Pi)$ be the associated symplectic foliation. The family of symplectic forms defined on the leaves of $\mathscr{S}$ yields an isomorphism $\mathscr{S} \simeq \mathscr{S}^*$. Consequently, $\mathscr{S}^*$ is a subbundle of TM, and $\Gamma(\ker \Pi)$ is an ideal in $\Omega^1(M)$ with respect to the Lie algebroid structure in T^*M given by the bracket (1). Moreover, $\mathscr{S}^*$ is a Lie subalgebroid of TM; its cohomology is called the *tangential Lichnerowicz–Poisson cohomology*. Now assume that M carries the action ρ of a Lie group G; if $\mathfrak{g}$ is the Lie algebra of G, then for every $\xi \in \mathfrak{g}$ by

$$\xi^* = \frac{d}{dt}\rho_{\exp(-t\xi)}\Big|_{t=0}$$

we denote the corresponding fundamental vector field. (Thus, we have the Lie algebra homomorphism $\tilde{\rho} \colon \mathfrak{g} \to \mathfrak{X}(M)$, $\tilde{\rho}(\xi) = \xi^*$.) If the G-action is tangent to $\mathscr{S}^*$, we obtain an infinitesimal action $b \colon \mathfrak{g} \to \Gamma(\mathscr{S}^*)$. (This is what is called a *cotangential lift* in [12].)

If, for some $\xi \in \mathfrak{g}$, a point $x \in M$ is a zero of ξ^*, then we have the usual endomorphism

$$L_{\xi,x} \colon T_pM \to T_pM, \qquad L_{\xi,x}(v) = [\xi^*, v].$$

Consider the graded vector space

$$\mathfrak{A}^\bullet = \mathrm{Sym}^\bullet(\mathfrak{g}^*) \otimes \Gamma(\wedge^\bullet A^*)$$

with grading

$$\deg(\mathscr{P} \otimes \beta) = 2\deg(\mathscr{P}) + \deg\beta,$$

where $\mathscr{P} \in \mathrm{Sym}^\bullet(\mathfrak{g}^*)$ and $\beta \in \Gamma(\wedge^\bullet A^*)$.

We treat $\mathscr{P}$ as a polynomial function on $\mathfrak{g}$ and define an equivariant differential $\delta_{\mathfrak{g}} \colon \mathfrak{A}^\bullet \to \mathfrak{A}^{\bullet+1}$ by setting

$$(\delta_{\mathfrak{g}}(\mathscr{P} \otimes \beta))(\xi) = \mathscr{P}(\xi)(\delta(\beta) - i_{b(\xi)}\beta), \tag{10}$$

where both sides have been evaluated on an element $\xi \in \mathfrak{g}$. If we set $\mathfrak{A}_G^\bullet = \ker \delta_{\mathfrak{g}}^2$, then $(\mathfrak{A}_G^\bullet, \delta_{\mathfrak{g}})$ is a complex, whose cohomology will be denoted by $H_G^\bullet(A)$ and called the *equivariant cohomology* of the Lie algebroid A (to be more precise, of the pair (A, b)).

By considering the graded vector space

$$\mathfrak{Q}^\bullet = \mathfrak{A}^\bullet \otimes \Gamma(Q_A) = \mathrm{Sym}^\bullet(\mathfrak{g}^*) \otimes \Gamma(\wedge^\bullet A^* \otimes Q_A)$$

with the differential $\tilde{\delta}_{\mathfrak{g}}$ obtained by coupling $\delta_{\mathfrak{g}}$ with the differential D, and by setting $\mathfrak{Q}_G^\bullet = \ker \tilde{\delta}_{\mathfrak{g}}^2$, one can also define the twisted equivariant cohomology $H_G^\bullet(Q_A)$, and there is a cup product

$$H_G^i(A) \otimes H_G^k(Q_A) \to H_G^{i+k}(Q_A).$$

Now let us write out a localization formula. In view of Proposition 2.7, its right-hand side can be computed by means of the usual localization formula in equivariant de Rham cohomology; the integral of an equivariantly closed $\mathfrak{Q}_G^\bullet$-cocycle $\gamma(\xi)$ is actually the integral of the differential form $p(\gamma(\xi))$, and $p(\gamma)$ is a cocycle in the equivariant de Rham complex by Proposition 2.7. It is indeed quite easy to prove the identity

$$p(\tilde{\delta}_{\mathfrak{g}}(\gamma)) = (-1)^k d_{\mathfrak{g}}(p(\gamma)), \tag{11}$$

where $\gamma \in \mathfrak{Q}_G^k$. Here $d_{\mathfrak{g}}$ is the differential in the usual equivariant de Rham cohomology. This follows from Proposition 2.7 and the relations

$$i_{\xi^*}p(\gamma) = i_{\xi^*}[a(\psi \lrcorner X) \lrcorner \mu] = (\xi^* \wedge a(\psi \lrcorner X)) \lrcorner \mu = a(b(\xi) \wedge (\psi \lrcorner X)) \lrcorner \mu$$
$$= (-1)^{k-1}a((i_{b(\xi)}\psi) \lrcorner X) \lrcorner \mu = (-1)^{k-1}p(i_{b(\xi)}\gamma)$$

if we set $\gamma = \psi \otimes X \otimes \mu$.

Let M be a closed manifold that carries an action ρ of a compact Lie group G. We also assume that M is oriented and that an element $\xi \in \mathfrak{g}$ has been chosen such that $\xi^* = \tilde{\rho}(\xi)$ (this has been defined in Example 3.1) has only isolated zeros. We denote the set of such zeros by M_ξ. Note that, owing to the compactness of G, we have $\det(L_{\xi,x}) \neq 0$ at every isolated zero x, and the dimension m of M is necessarily even (as we shall assume henceforth).

If the rank r of A is smaller than the dimension m of M, then for every equivariantly closed $\gamma \in \mathfrak{Q}^\bullet$ we have $\int_M \gamma(\xi) = 0$ for dimensional reasons, since $p(\gamma)_0 = 0$ in that case. (Here the subscript 0 denotes the piece of degree 0 in the usual de Rham grading.) We can therefore assume in what follows that $r \geqslant m$.

Theorem 3.2. *Let M be a closed oriented m-dimensional manifold on which a compact Lie group G acts. Let A be a rank r Lie algebroid on M, where $r \geqslant m$, and assume that there exists a Lie algebra homomorphism $b\colon \mathfrak{g} \to \Gamma(A)$ such that the diagram*

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\ b\ } & \Gamma(A) \\
& {\scriptstyle \tilde{\rho}} \searrow & \downarrow {\scriptstyle a} \\
& & \mathfrak{X}(M)
\end{array}
\tag{12}
$$

commutes; in other words, $\tilde{\rho}$, $\tilde{\rho}(\xi) = \xi^*$, is a Lie algebra homomorphism. (Here $\mathfrak{g}$ is the Lie algebra of G.) Moreover, assume that $\xi \in \mathfrak{g}$ and the associated fundamental vector field ξ^* has only isolated zeros. Finally, let $\gamma \in \mathfrak{Q}^\bullet$ be equivariantly closed, $\tilde{\delta}_\mathfrak{g}\gamma = 0$.

Then the following localization formula holds:

$$
\int_M \gamma(\xi) = (-2\pi)^{m/2} \sum_{x \in M_\xi} \frac{p(\gamma(\xi))_0(x)}{\det^{1/2} L_{\xi,x}}.
\tag{13}
$$

Proof. Since on the left-hand side we actually integrate the conventional differential form $p(\gamma(\xi))$, we see that the formula follows from identity (11) and the usual localization formula. $\square$

One can readily verify that in the case of the "trivial" algebroid given by the tangent bundle with the identity map as anchor, this formula is reduced to the ordinary localization formula for equivariant de Rham cohomology (e.g., see [2]).

Remark 3.3. If $r \geqslant m$ and the rank of the linear morphism a at the point p is not maximal (i.e., is less than m), then $p(\gamma(\xi))_0(x) = 0$.

Remark 3.4. As a special case of Theorem 3.2, one can state a localization formula related to the action of a vector field on M. Let M be a compact oriented m-dimensional manifold, and let $X \in \Gamma(\mathrm{T}M)$ be a vector field with isolated zeros on M that generates a circle action. Let A be a rank r Lie algebroid on M such that there exists an $\tilde{X} \in \Gamma(A)$ with $a(\tilde{X}) = X$ (where a is the anchor map). Then, for the integration of a form $\gamma \in \Gamma(\wedge^\bullet A^* \otimes \wedge^r A \otimes \wedge^m T^*M)$ such that $\tilde{\delta}_X \gamma := (\tilde{\delta} - i_{\tilde{X}})\gamma = 0$, the localization formula (3.2) holds.

One can replace the assumption that X generates a circle action by assuming that X is an isometry of a Riemannian metric on M. $\square$

4. Connections and Characteristic Classes of Lie Algebroids

Several applications of the localization formula can be given by using the notion of characteristic class of a Lie algebroid. We start by introducing the concept of A-*connection*, see [17].

Let A be a Lie algebroid with anchor a, and let $P \xrightarrow{p} M$ be a principal bundle with structure group K. Note that the pullback $p^*A = A \times_M P$ carries a natural K-action, and $A \simeq p^*A/K$. The tangent bundle TP carries a natural K-action as well, and one has $p_*(vk) = p_*(v)$ for $v \in TP$ and $k \in K$; consequently, we have the induced map $p_*\colon TP/K \to TM$, which is the anchor of the Atiyah algebroid associated with P, see (2).

Definition 4.1. An *A-connection* on P is a bundle map $\eta\colon p^*A \to TP$ such that
(1) The diagram

$$
\begin{array}{ccc}
p^*A & \xrightarrow{\ \eta\ } & TP \\
\downarrow & & \downarrow{\scriptstyle p_*} \\
A & \xrightarrow{\ a\ } & TM
\end{array}
$$

commutes.

(2) η is K-equivariant; i.e., $\eta(uk, \alpha) = R_{k*}\eta(u, \alpha)$ for all $k \in K$, $u \in P$, and $\alpha \in A$. (Here R_k is the structural right action of an element $k \in K$ on P.)

If P is the bundle of linear frames of A, then η is called an *A-linear connection*.

Being equivariant, an A-connection η on a principal K-bundle P defines a bundle map $\omega_\eta\colon A \to TP/K$, called the 1-*section of the connection* η. One has $p_* \circ \omega_\eta = a$.

Remark 4.2. 1. The usual notion of connection is recovered by taking $A = TM$, and η is then the corresponding horizontal lift $\eta\colon p^*TM \to TP$.

2. An ordinary connection on P (regarded as the associated horizontal lift $\zeta\colon p^*TM \to TP$) defines an A-connection η on P by the formula $\eta = \zeta \circ p^*a$.

If $E \xrightarrow{p_E} M$ is a vector bundle associated with P via a representation of K on a linear space, then an A-connection on P defines a similar structure on E, that is, a bundle map $\eta_E\colon p_E^*A \to TE$ that makes the diagram

$$
\begin{array}{ccc}
p_E^*A & \xrightarrow{\ \eta_E\ } & TE \\
{\scriptstyle A}\downarrow & & \downarrow{\scriptstyle p_{E*}} \\
A & \xrightarrow{\ a\ } & TM
\end{array}
$$

commute. The A-connection η_E defines a covariant derivative $\nabla\colon \Gamma(A) \otimes_\mathbb{R} \Gamma(E) \to \Gamma(E)$ in the usual way: if $\phi\colon TP/K \to \mathrm{Diff}_0^1(E)$ is the natural map,[*] one sets $\nabla_\alpha = (\phi \circ \omega_\eta)(\alpha)$. This covariant derivative satisfies the Leibniz rule

$$
\nabla_\alpha(fs) = f\nabla_\alpha(s) + a(\alpha)(f)s \tag{14}
$$

for all functions f on M.

Let us introduce the notion of *G-equivariant A-connection*. Assume that a Lie group G acts on M and that this action ρ lifts to an action $\hat{\rho}$ on A; this means that for every $g \in G$ we have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \hat{\rho}_g\ } & A \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
TM & \xrightarrow{\ \rho_{g*}\ } & TM
\end{array}
$$

Moreover, we assume that ρ also lifts to an action $\tilde{\rho}$ on the principal K-bundle P commuting with the structural K-action.

Definition 4.3. A *G-equivariant A-connection* η *on* P is an A-connection η such that the diagram

$$
\begin{array}{ccc}
p^*A & \xrightarrow{\ \hat{\rho}_g\ } & p^*A \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\
TP & \xrightarrow{\ \tilde{\rho}_{g*}\ } & TP
\end{array}
$$

commutes for every $g \in G$.

---

[*] A section X of TP/K is a K-invariant vector field on P. Since there is an obvious map $TP \to \mathrm{Diff}_0^1(P \times V)$, where V is the standard fibre of E, by equivariance X yields a differential operator on E with scalar symbol.

Since the action of G on P commutes with the action of K, we have an induced action $\tilde{\rho}_*$ on TP/K, and the condition for η to be G-equivariant can be stated in terms of the connection 1-section ω_η as the commutativity of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\hat{\rho}_g} & A \\
{\scriptstyle \omega_\eta}\downarrow & & \downarrow{\scriptstyle \omega_\eta} \\
TP/K & \xrightarrow{\tilde{\rho}_{g*}} & TP/K
\end{array}
$$

The *curvature* $\mathscr{R}_\eta$ of an A-connection η on a principal K-bundle P can be defined in terms of the map ω_η as the element in $\Gamma(\Lambda^2 A^* \otimes TP/K)$ given by

$$\mathscr{R}_\eta(\alpha, \beta) = [\omega_\eta(\alpha), \omega_\eta(\beta)] - \omega_\eta(\{\alpha, \beta\}).$$

By construction, $p_* \circ \mathscr{R}_\eta = 0$, so that $\mathscr{R}_\eta$ is an element in $\Gamma(\Lambda^2 A^* \otimes \mathrm{ad}(P))$. The curvature $\mathscr{R}_\eta$ satisfies an analog of the *structure equations* and *Bianchi* identities. These identities are conveniently stated in terms of the so-called *exterior A-derivative*

$$D_A \colon \Gamma(\wedge^\bullet A^* \otimes TP/K) \to \Gamma(\wedge^{\bullet+1} A^* \otimes TP/K),$$

$$(D_A \chi)(\alpha_1, \ldots, \alpha_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} [\omega_\eta(\alpha_i), \chi(\alpha_1 \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1})]$$

$$+ \sum_{i<j} (-1)^{i+j} \chi(\{\alpha_i, \alpha_j\}, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{p+1}),$$

in the form

$$\mathscr{R}_\eta = D_A \omega_\eta - \tfrac{1}{2}[\omega_\eta, \omega_\eta], \qquad D_A \mathscr{R}_\eta = 0. \tag{15}$$

Note that the equivariance of the connection can be expressed by the condition

$$[D_A, \mathscr{L}_{\tilde{\xi}^*}] = 0$$

for all $\xi \in \mathfrak{g}$. Here $\mathscr{L}_{\tilde{\xi}^*}$ is the Lie derivative of sections of TP/K with respect to the section $\tilde{\xi}^*$ of TP/K induced by the vector field on P that generates the action of G (i.e., $\mathscr{L}_{\tilde{\xi}^*}(v) = [\tilde{\xi}^*, v]$).

If the connection η is equivariant, we can consider the equivariant version of these relations, in particular, by defining the equivariant exterior A-derivative $D_A^{\mathfrak{g}}$ that acts on $\mathrm{Sym}^\bullet(\mathfrak{g}^*) \otimes \Gamma(\wedge^\bullet A^* \otimes TP/K)$ as

$$(D_A^{\mathfrak{g}} \chi)(\xi) = D_A(\chi(\xi)) - (i_{\tilde{\xi}^*} \otimes \mathrm{id})(\chi(\xi)).$$

Moreover, we define the equivariant curvature $R_\eta^{\mathfrak{g}}$ of η as

$$(R_\eta^{\mathfrak{g}} \chi)(\xi) = R_\eta(\chi(\xi)) + \mathscr{L}_{\tilde{\xi}^*}(\chi(\xi)) - [D_A, i_{\tilde{\xi}^*} \otimes \mathrm{id}](\chi(\xi)) = R_\eta(\chi(\xi)) + \mu(\chi(\xi)),$$

where the last equality defines the "moment map" μ. Furthermore, the square brackets in this equation stand for the anticommutator. An easy computation shows that the equivariant curvature satisfies the equivariant Bianchi identity

$$D_A^{\mathfrak{g}} R_\eta^{\mathfrak{g}} = 0. \tag{16}$$

We can also write out identities (15) and (16) in local form involving the *local connection 1-sections* defined as follows. Let $\{U_i\}$ be an open cover of M over which the bundle P trivializes. Then one has local isomorphisms

$$\psi_j \colon (TP/K)|_{U_j} \to TU_j \times \mathfrak{k},$$

where $\mathfrak{k}$ is the Lie algebra of K. The local connection 1-sections are defined by the condition

$$\omega_j(\alpha) = \mathrm{pr}_2 \circ \psi_j \circ \omega_\eta(\alpha)$$

(where pr_2 is the projection onto the second factor in $TU_j \times \mathfrak{k}$), and one defines the local curvature 2-sections $\mathscr{R}_j$ in a similar way. Identities (15) acquire the form

$$\mathscr{R}_j = \delta\omega_j + \tfrac{1}{2}[\omega_j, \omega_j], \qquad \delta\mathscr{R}_j + [\omega_j, \mathscr{R}_j] = 0. \tag{17}$$

In the same way, the equivariant curvature can be represented by local 2-sections $\mathscr{R}_j^{\mathfrak{g}}$ which, in view of Eq. (17), satisfy the identities

$$\delta_{\mathfrak{g}} \mathscr{R}_j^{\mathfrak{g}} + [\omega_j, \mathscr{R}_j^{\mathfrak{g}}] = 0. \tag{18}$$

The Chern–Weil homomorphism is defined as follows. Let $I^\bullet(\mathfrak{k}) = (\mathrm{Sym}^\bullet \mathfrak{k}^*)^K$. We choose an A-connection η for P and define an element $\lambda_Q \in C_A^{2\ell}$,

$$\lambda_Q(\alpha_1, \ldots, \alpha_{2\ell}) = \sum_\sigma (-1)^\sigma \widetilde{Q}(\mathscr{R}_\eta(\alpha_{\sigma_1}, \alpha_{\sigma_2}), \ldots, \mathscr{R}_\eta(\alpha_{\sigma_{2\ell-1}}, \alpha_{\sigma_{2\ell}})),$$

for any polynomial $Q \in I^\ell(\mathfrak{k})$ of degree ℓ, where the summation runs over all permutations of 2ℓ objects and $\widetilde{Q}$ is the polarization of Q, i.e., the unique Ad-invariant symmetric function of ℓ variables in $\mathfrak{k}$ such that $\widetilde{Q}(\chi, \ldots, \chi) = Q(\chi)$ for all $\chi \in \mathfrak{k}$. One proves that this cochain is δ-closed and that the resulting cohomology class $[\lambda_Q]$ does not depend on the connection, thus defining a graded ring homomorphism $\lambda \colon I^\bullet(\mathfrak{k}) \to H^{2\bullet}(A)$. If $\tilde{\lambda} \colon I^\bullet(\mathfrak{k}) \to H_{dR}^{2\bullet}(M)$ is the usual Chern–Weil homomorphism into the de Rham cohomology of M, then there is a commutative diagram

$$\begin{array}{ccc} I^\bullet(\mathfrak{k}) & \xrightarrow{\tilde{\lambda}} & H_{dR}^{2\bullet}(M) \\ & \searrow{\scriptstyle\lambda} & \downarrow{\scriptstyle a^*} \\ & & H^{2\bullet}(A) \end{array} \tag{19}$$

Using the Chern–Weil homomorphism, one can introduce various sorts of characteristic classes for the Lie algebroid A. However, owing to diagram (19) (and somehow unpleasantly) these characteristic classes are none other than the image under a^* of the corresponding characteristic classes of the bundle E. To show this, choose any (ordinary) connection on P to compute a characteristic class in $H_{dR}^\bullet(M)$ and the A-connection associated with it (see Remark 4.2 (2)) to compute a characteristic class in $H^\bullet(A)$. The two characteristic classes are then related by the morphism a^* in diagram (19).

In the following, we use "Pontryagin type" characteristic classes: namely, we take $\mathfrak{k} = \mathrm{gl}(r, \mathbb{C})$, so that P is the bundle of linear frames of a complex vector bundle E. We assume that E is the complexification of a real vector bundle. Let Q_i be the ith elementary Ad-invariant polynomial, and let λ_i be the corresponding characteristic class. These characteristic classes vanish whenever i is odd. (To verify this, take a connection on P compatible with a fibre metric on E and compute the characteristic classes by means of the induced A-connection.)

If $Q \in I^\bullet(\mathrm{gl}(r, \mathbb{C}))$ is an Ad-invariant homogeneous symmetric polynomial of degree $2i$ on the Lie algebra $\mathrm{gl}(r, \mathbb{C})$, then one has $Q(\mathscr{R}_\eta^{\mathfrak{g}}) \in \mathfrak{A}_G^{4i}$. The following statement is easy to prove.

Proposition 4.4. *The element* $Q(\mathscr{R}_\eta^{\mathfrak{g}})$ *is* $\delta_{\mathfrak{g}}$*-closed. The corresponding cohomology class* $\lambda_Q^{\mathfrak{g}}(A)$ $\in H_G^{4i}(A)$ *does not depend on the equivariant connection* η.

Proof. First, one shows that the element $Q(\mathscr{R}_\eta^{\mathfrak{g}})$ is $\delta_{\mathfrak{g}}$-closed by using (18). To prove the second assertion, note that if η and η' are two equivariant A-connections on the principal bundle P, then one can define the one-parameter family of connections

$$\eta_t = t\eta' + (1-t)\eta$$

with $0 \leqslant t \leqslant 1$. Set

$$q(\eta, \eta')(\alpha_1, \ldots, \alpha_{2i}) = i \sum_\sigma (-1)^\sigma \int_0^1 \widetilde{Q}\bigg(\frac{d}{dt}\omega_{\eta_t}(\alpha_{\sigma_1}, \alpha_{\sigma_2}), \mathscr{R}_{\eta_t}^{\mathfrak{g}}(\alpha_{\sigma_3}, \alpha_{\sigma_4}), \ldots, \mathscr{R}_{\eta_t}^{\mathfrak{g}}(\alpha_{\sigma_{2i-1}}, \alpha_{\sigma_{2i}})\bigg) dt,$$

where $\widetilde{Q}$ is the polarization of the polynomial Q. Now a straightforward computation using identity (18) and the formula

$$\frac{d}{dt}\mathscr{R}_{\eta_t}^{\mathfrak{g}} = D_A^{\mathfrak{g}}\frac{d}{dt}\omega_{\eta_t}$$

shows that

$$Q(\mathscr{R}_{\eta'}^{\mathfrak{g}}) - Q(\mathscr{R}_\eta^{\mathfrak{g}}) = \delta_{\mathfrak{g}}q(\eta, \eta')$$

whence the assertion follows. □

If Q is the standard ith elementary Ad-invariant polynomial ς_i on $\mathrm{gl}(r, \mathbb{C})$, then we denote by $\lambda_i^{\mathfrak{g}}(A)$ $(i = 1, \ldots, r)$ the corresponding equivariant characteristic class (vanishing for odd i). As was discussed previously, these are just the images under the morphism a^* (the adjoint of the anchor map) of the equivariant Chern classes of the (complexified) vector bundle E.

5. Bott Formula

By way of application of our localization theorem, we prove a result that generalizes the classical Bott formula [3] as well as similar results due to Cenkl [9] and Kubarski [15]. The Bott formula comes in different flavors according to the assumptions that one makes on the vector field occurring in the formula. The case we consider here extends the usual Bott formula for a vector field that generates a circle action and has isolated critical points.

Let Φ be a monomial in $n = [r/4]$ variables. By W_Φ we denote its *total weight* defined by assigning weight $4i$ to the ith variable. We use the monomial Φ to assign a real number to the algebroid A. Assume that $W_\Phi \leqslant r$, and let $\Xi^{\mathfrak{g}}$ be an equivariant twisted cocycle of degree $r - W_\Phi$; i.e., $\Xi^{\mathfrak{g}} \in \mathfrak{Q}_G^{r-W_\Phi}$. Let $\Xi = \Xi^{\mathfrak{g}}(0) \in \widetilde{C}_A^{r-W_\Phi}$ be the zero degree term in the cocycle $\Xi^{\mathfrak{g}}$ treated as a function of $\xi \in \mathfrak{g}$; it satisfies $\tilde{\delta}\Xi = 0$. We can define

$$\Phi_\Xi(A) = (-2\pi)^{-m/2}\int_M \Phi(\lambda_2(A), \ldots, \lambda_{2n}(A)) \wedge \Xi,$$

where the λ_i are the characteristic classes of the vector bundle A as defined in the previous section. This number depends only on the Lie algebroid A and the cohomology class Ξ. We also define an element in $\mathrm{Sym}^{\bullet}(\mathfrak{g}^*)$ by the formula

$$\Phi_\Xi^{\mathfrak{g}}(A) = (-2\pi)^{-m/2}\int_M \Phi(\lambda_2^{\mathfrak{g}}(A), \ldots, \lambda_{2n}^{\mathfrak{g}}(A)) \wedge \Xi^{\mathfrak{g}}$$

$$= (-2\pi)^{-m/2}\int_M \Phi(\varsigma_2(\mathscr{R}_\eta + \mu), \ldots, \varsigma_{2n}(\mathscr{R}_\eta + \mu)) \wedge \Xi^{\mathfrak{g}}. \tag{20}$$

Of course, $\Phi_\Xi(A) = \Phi_\Xi^{\mathfrak{g}}(A)(0)$. One has the following result, in the spirit of the Bott formula, which readily follows from Theorem 3.2.

Theorem 5.1. *Let A be a rank r Lie algebroid on an m-dimensional compact oriented manifold M, and let $\alpha \in \Gamma(A)$ be any section that generates a circle action on A and satisfies the condition that $a(\alpha)$ has isolated zeros. Let Φ be a polynomial in $n = [r/4]$ variables whose monomials have total weight W_Φ. If $r \geqslant m \geqslant W_\phi$, then*

$$\Phi_\Xi(A) = \sum_{x \in M_{a(\alpha)}} \frac{\Phi(c_2(L_{a(\alpha),x}), \ldots, c_{2n}(L_{a(\alpha),x}))\, p(\Xi^{\mathfrak{g}})_0}{\det^{1/2} L_{a(\alpha),x}}, \tag{21}$$

where the classes $c_i(L_{a(\alpha),x})$ are the equivariant Chern classes of the endomorphism $L_{a(\alpha),x}$ acting on the tangent space $T_x M$ at a zero x of $a(\alpha)$ (see [3]).

Proof. The right-hand side of (21) can be obtained from the right-hand side of (13) if we take into account two facts. First, we can evaluate the equivariant characteristic classes occurring in $\Phi_\Xi^{\mathfrak{g}}(A)$ by choosing an equivariant A-connection induced in the principal bundle $GL(A)$ by an ordinary equivariant connection ζ on the vector bundle A. This way, we have

$$\Phi(\lambda_2^{\mathfrak{g}}(A), \ldots, \lambda_{2n}^{\mathfrak{g}}(A)) = a^*(\Phi(\nu_2^{\mathfrak{g}}(A), \ldots, \nu_{2n}^{\mathfrak{g}}(A))),$$

27

where the classes $\nu_i^{\mathfrak{g}}$ are the equivariant Chern classes of the complexification of the vector bundle A. Second, if $R_\zeta^{\mathfrak{g}}$ is the equivariant curvature of ζ, then

$$(\varsigma_i(R_\zeta^{\mathfrak{g}}))_0(x) = c_i(L_{a(\alpha),x})$$

for every symmetric elementary function ς_i and every zero x of $a(\alpha)$.

For $A = TM$, formula (21) is reduced to the ordinary Bott formula. $\qquad\square$

6. Concluding Remarks

As we shall show in the forthcoming paper [5], Theorem 5.1 generalizes several localization formulas associated with the action of a holomorphic vector field on a complex manifold which can be equivariantly lifted to an action on a holomorphic vector bundle ([1], [10], [8]) and, in particular, reproduces Grothendieck's residue theorem.

On the other hand, our formula can be generalized in several directions. One of these would be a localization formula for equivariant cohomology associated with the action of a Lie group on a Courant algebroid. This should encompass several formulas that recently appeared in the literature and are mostly concerned with generalized Calabi–Yau structures ([6], [20], [21], [13]) and should reproduce our formula if the Courant algebroid is reduced to a Lie algebroid.

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