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Stabilization using in-domain actuator: a numerical method for a non linear parabolic partial differential equation

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Abstract. This paper deals with the problem of null controllability for an unstable nonlinear parabolic partial differential equation (PDE) system considering in-domain actuator. The main objective of this communication is to provide an efficient control law in order to stabilize the system state close to zero in a desired time whatever the initial state is. A numerical approach is developed and in order to highlight the relevance of the proposed control strategy, a realistic physical problem is investigated. Thermal evolution of a thin rod with homogeneous Dirichlet boundaries conditions is considered. Thermal state is described by the heat equation and assuming that thermal conductivity is temperature dependent, a nonlinear mathematical model has to be taken into account. Considering that all the model inputs are known, a direct problem is numerically solved (regarding a finite element method) in order to estimate the temperature at each point of the 1D geometry and at each instant. Then an inverse problem is formulated in such a way as to determine the in-domain control which ensures a final temperature close to zero. An iterative regularization method based on the conjugate gradient method (CGM) is developed for the minimization of a quadratic cost function (output error). Several numerical experimentations are provided in order to discuss the numerical approach attractiveness.

Keywords: Parabolic partial differential equation · Inverse problem · Conjugate gradient method.

1 Introduction

Evolution of numerous physical phenomena could be described by a mathematical model based on a set of PDE: wave equation, Navier-Stokes equations that describe the motion of viscous fluid substances, Saint-Venant equations that describe the flow below a pressure surface in a fluid in one direction, heat equation

that describes the temperature evolution ... The last one is issued from energy balance (Fourier's Law) and leads to a set of a parabolic PDE with initial and boundaries conditions. Control and stabilization of physical processes described by PDE systems require multidisciplinary study and are one of the most extensive research of the past decades [1].

Recent developments could be mentioned in [2-10] and from the theoretical point of view the investigated mathematical models are usually linear. However, in numerous realistic problems non linearities could not be neglected and development of numerical approaches in order to build relevant control law is a crucial requirement. In the specific case of nuclear fusion for example, the control of the spatial distribution of the safety profile in tokamak plasma is highly complex [11-12] and should be investigated carefully by taken into account non linearities and coupled phenomena (magnetohydrodynamics equation and heat equations).

In the following, our purpose is to stabilize in finite time the spatial distribution of the solution of a parabolic PDE system near to the stable desired state. The main challenge is to identify the in-domain flux (heating or cooling) which ensures the null controllability. It is well known that such inverse heat conduction problem (IHCP) is ill posed in Hadamard sense [13]. In the framework of thermal properties identification, the iterative regularization method based on the CGM has been developed in [14]. In recent works, authors have proposed new developments for mobile heating source tracking in 2D and 3D geometries [15-17] and for quasi online identification of a temperature dependent characteristics in [18].

The paper is organized as follows. Next section is devoted to the mathematical model of the physical system: direct problem is defined in order to describe the heat conduction in the investigated domain (thermal characteristics of the heated material are also given). A finite element method implemented with Comsol Multiphysics Solver and Matlab software is used for numerical simulation [19-21]. Several numerical configurations are tested in order to highlight the instability of the PDE system. In the third section, identification of the control law is formulated as a minimization problem and the optimization method is based on iterative descent scheme. This approach is based on iterative numerical resolution of three well-posed problems: direct problem, sensitivity problem and adjoint problem. Considering the constructive method of control proposed in the third section, several numerical configurations are tested in the fourth section. Finally, concluding remarks and outlooks are discussed in the last section.

2 Studied Configuration

2.1 Problem Statement

In this section, the investigated thermal system is briefly presented. Let us study a thin plate. Considering that the plate thickness is neglected (temperatures are equal on the upper and on the lower face of the plate), each point of the domain is defined by its space variables $(x, y) \in [0, L] \times \mathbb{R}$. The time variable is $t \in T = [0, t_f]$ where t_f is the final time. The in-domain actuator location is centered

on $0 < a < L$ and the control $u_a(x, t)$ acts on $\omega = [a - \varepsilon, a + \varepsilon] \subset \Omega = [0, L]$ (see Fig. 1). Such localized internal control can be encountered in the context of nuclear fusion for a current controlled by microwaves, for example [22-23].

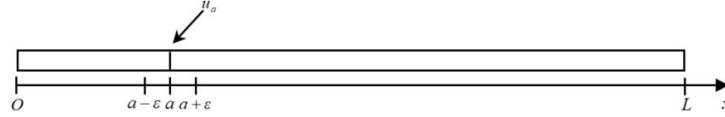


Fig. 1. Studied geometry.

The initial temperature distribution at $t = 0$ is denoted by $f(x)$. Let us denote by $\theta(x, t)$ the temperature in K at point x and time t . On the both boundaries $\{x = 0; x = L\}$, homogeneous Dirichlet condition is considered. Then, heat exchanges on the two faces of the plate could be described by the following system:

$$\left\{ \begin{array}{ll} \rho C \frac{\partial \theta(\cdot)}{\partial t} - \frac{\partial}{\partial x} \left(\lambda(\theta) \frac{\partial \theta(\cdot)}{\partial x} \right) = \xi \theta(\cdot) + u_a(\cdot) & \forall (x, t) \in \Omega \times T \\ \theta(0, t) = \theta(L, t) = 0 & \forall t \in T \\ \theta(x, 0) = f(x) & \forall x \in \Omega \end{array} \right. \quad (1)$$

The first equation of system (1) is the parabolic PDE describing the evolution of the temperature distribution where: ρC is the volumic heat in $J.m^{-3}.K^{-1}$ and λ is the thermal conductivity in $W.m^{-1}.K^{-1}$. The non linearity of this heat equation is due to the thermal dependence of the thermal conductivity $\lambda(\theta)$. The coefficient ξ in $W.m^{-3}.K^{-1}$ can be negative or positive and $\xi \theta(x, t)$ describes the heat transfer with the surrounding and affects the system stability. Control $u_a(x, t)$ in $W.m^{-3}$ acts on ω and aims to stabilize the temperature such as $\theta(x, t_f) = 0, \forall x \in \Omega$. If all the input parameters of (1) are known, this system defines a well posed direct problem and its resolution leads to the determination of the temperature distribution evolution $\theta(x, t), \forall (x, t) \in \Omega \times T$.

2.2 Direct problem Resolution

In the following example, realistic input parameters are taken into account: $L = 0.1m, \rho C = 10^6 J.m^{-3}.K^{-1}$ and initial temperature is defined as follows:

$$f(x) = 25 \left[\frac{1}{2} \exp \left(\frac{-(x - 0.02)^2}{5 \times 10^{-5}} \right) + \exp \left(\frac{-(x - 0.05)^2}{10^{-4}} \right) \right].$$

Thermal conductivity is assumed to be: $\lambda(\theta) = 33 + \exp \left(\frac{\theta}{40} \right)$. Without any internal control ($u_a(x, t) = 0$), if $\xi = 0$ the temperature naturally converges towards 0 and if $\xi < 0$, the term $\xi \theta(x, t)$ increases the convergence speed. It is well known that there exists a threshold K such that if $\xi > K > 0$, the system is not stable and $\lim_{t \rightarrow \infty} \|\theta(t)\|_{L^2(\Omega)}^2 = \lim_{t \rightarrow \infty} \int_0^L [\theta(x, t)]^2 dx = +\infty$.

Effect of different values of ξ are taken into account and direct problem (1) is solved until $t_f = 100s$. Temperature are determined using a software based on a finite element method (Comsol-Multiphysics solver interfaced with Matlab program). Temperature norms $\|\theta(t)\|_{L^2(\Omega)}^2$ for several values of parameter ξ are shown in Fig. 2. Considering Fig. 2, it is obvious that according to the first

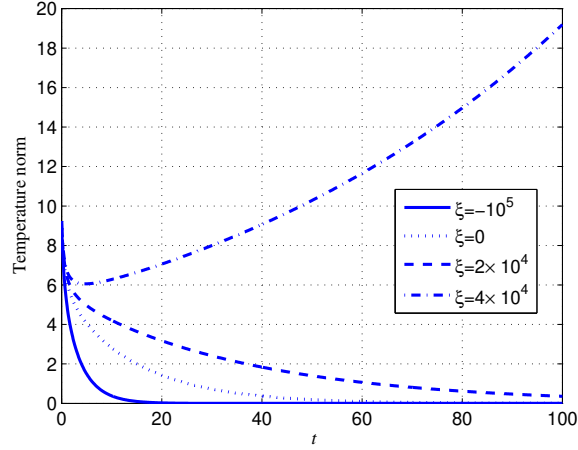


Fig. 2. Evolution of temperature norm.

three small values of ξ , system (1) is stable; without control, temperature norm decreases and temperature converges towards zero in the domain. For $\xi = 4 \times 10^4$, system (1) is unstable; $\xi\theta$ brings a non-stability that tends to move the system away from 0. Such behaviour is shown in Fig. 3.

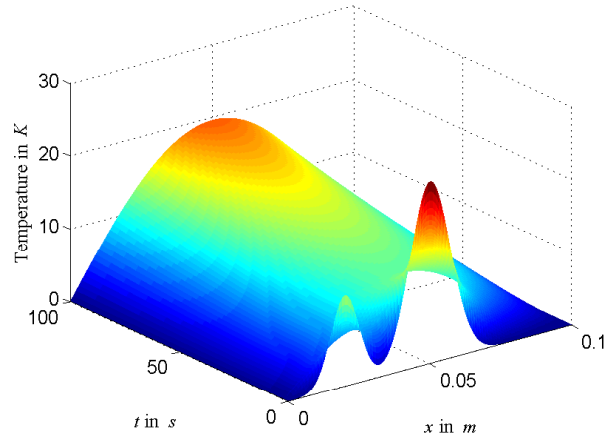


Fig. 3. Temperature evolution - $\xi = 4 \times 10^4$.

In the following section, unstable configuration where $\xi = 4 \times 10^4$ is considered and an inverse problem is proposed in order to identify the unknown control law $u_a(x, t)$ which will be able to stabilize system (1).

3 Inverse Problem

The aim of the following inverse problem is to identify an in-domain control $u_a(\cdot)$ such that $\|\theta(x, t_f)\|_{L^2(\Omega)}^2 \approx 0$. An iterative minimization method is developed in order to minimize at each iteration k the quadratic cost function:

$$J(\theta^k) = \frac{1}{2} \int_0^L [\theta^k(x, t_f)]^2 dx \quad (2)$$

A discrete formulation could be considered and leads to a similar minimization of $\|\theta(x, t_f)\|_{L^2(\Omega)}^2$:

$$J_{discrete} = \sum_{i=1}^N [\hat{\theta}_i(t_f)]^2, \forall i = 1, \dots, N$$

where N is the number of sensors and $\hat{\theta}_i$ is the measured temperature at the sensor location x_i .

An iterative algorithm based on CGM is implemented in order to solve this inverse problem written as a minimization one. CGM is a descent method that solves the problem of parametric identification by stopping the minimization when a relevant threshold J_{stop} is obtained. Such method has been developed for thermal applications in [14-18]. At each iteration k of the algorithm, three well-posed problems have to be solved:

1. The direct problem in order to determine the temperature distribution $\theta^k(x, t_f)$, and then to estimate the criterion $J(\theta^k)$,
2. The adjoint problem to determine the gradient of the cost function $J(\theta^k)$ and thus to define the next descent direction d^k ,
3. The sensitivity problem to estimate the descent depth γ^k (in the descent direction).

Let us consider that the control $u_a^k(x, t)$ is a piecewise linear continuous function depending on time and space and can be formulated as follows:

$$u_a(x, t) = \sum_{j=1}^M u_{aj}(x) s_j(t)$$

where

$$u_{aj}(x) = \begin{cases} \frac{x-(a-\varepsilon)}{\varepsilon} u_{aj} & \text{if } x \in [a - \varepsilon, a] \\ \frac{(a+\varepsilon)-x}{\varepsilon} u_{aj} & \text{if } x \in [a, a + \varepsilon] \\ 0 & \text{if not} \end{cases}$$

and

$$s_j(t) = \begin{cases} \frac{(M+1)t - (j-1)t_f}{t_f} & \text{if } t \in \left[\frac{j-1}{M+1}t_f, \frac{j}{M+1}t_f \right] \\ \frac{(j+1)t_f - (M+1)t}{t_f} & \text{if } t \in \left[\frac{j}{M+1}t_f, \frac{j+1}{M+1}t_f \right] \\ 0 & \text{if not} \end{cases}$$

Functions $s_j(t)$ are the ‘‘hat functions’’, basis for piecewise linear functions. According to this notation $u_a^k(x, 0) = u_a^k(x, t_f) = 0$, a pointwise actuator could be investigated if ε is close to zero.

In the following three subsections, sensitivity problem, adjoint problem and a conjugate gradient algorithm are presented.

3.1 Sensitivity Problem

In order to calculate at iteration k the descent depth γ^k in the descent direction \mathbf{d}^k , the sensitivity problem has to be solved. Let us consider temperature variation $\theta(x, t) + \varepsilon_0 \delta\theta(x, t)$ induced by a variation of the thermal flux $u_a(x, t) + \varepsilon_0 \delta u_a(x, t)$. Formulation of system (1) satisfied by the temperature $\theta^+ = \theta + \varepsilon_0 \delta\theta$ and considering the control $u_a^+ = u_a + \varepsilon_0 \delta u_a$ leads to the following system while $\varepsilon_0 \rightarrow 0$. Sensitivity function $\delta\theta$ is solution of this so-called sensitivity problem:

$$\begin{cases} \rho C \frac{\partial \delta\theta^k(\cdot)}{\partial t} - \frac{\partial^2 (\lambda(\theta) \delta\theta^k(\cdot))}{\partial x^2} = \xi \delta\theta^k(\cdot) + \delta u_a^k(\cdot) & \forall (x, t) \in \Omega \times T \\ \delta\theta^k(0, t) = \delta\theta^k(L, t) = 0 & \forall t \in T \\ \delta\theta^k(x, 0) = 0 & \forall x \in \Omega \end{cases} \quad (3)$$

In this study, the variation of thermal flux provided by the actuator is:

$$\delta u_a(x, t) = \sum_{j=1}^M \delta u_{aj}(x) s_j(t).$$

At each iteration k , the identified control u_{aj}^{k+1} , $\forall j = 1, \dots, M$ is given by:

$$u_{aj}^{k+1} = u_{aj}^k - \gamma^k d_j^k \quad (4)$$

and γ^k is the optimal depth determined as follow:

$$\gamma^k = \arg \min_{\gamma \in \mathbf{R}} J(\mathbf{u}_a^k - \gamma \mathbf{d}^k) = \frac{- \int_0^L \theta^k(x, t_f) \delta\theta^k(x, t_f) dx}{\int_0^L [\delta\theta^k(x, t_f)]^2 dx} \quad (5)$$

where $\theta^k(x, t_f)$ is the solution of the direct problem (1) and $\delta\theta^k(x, t_f)$ is the solution of the sensitivity problem (3) (solved in descent direction \mathbf{d}^k).

In the following section adjoint problem is developed in order to determine the descent direction \mathbf{d}^k .

3.2 Adjoint Problem

In order to calculate at each iteration k , the gradient $\frac{\partial J(\theta^k)}{\partial u_{a_j}^k}$ and the descent direction \mathbf{d}^k , an adjoint problem is formulated. Let us denote by ℓ a Lagrangian formulation which is a function of $u_a^k(\cdot)$, $\theta^k(\cdot)$ and $\psi^k(\cdot)$ where $\psi^k(\cdot)$ is the adjoint function, fixed such as: $\frac{\partial \ell}{\partial \theta^k} \delta \theta^k = 0$ and $\frac{\partial \ell}{\partial \psi^k} \delta \psi^k = 0$. Furthermore, if $\theta(\cdot)$ is solution of system (1) then:

$$\ell(u_a^k, \theta^k, \psi^k) = J(\theta^k) \Rightarrow \delta \ell(u_a^k, \theta^k, \psi^k) = \delta J(\theta^k)$$

and variation of Lagrangian

$$\begin{aligned} \delta \ell(u_a^k, \theta^k, \psi^k) &= \int_0^L \theta^k(x, t_f) \delta \theta^k(x, t_f) dx \\ &+ \int_0^{t_f} \int_0^L \left[\rho C \frac{\partial \delta \theta^k}{\partial t} - \frac{\partial^2}{\partial x^2} (\lambda(\theta) \delta \theta^k) - \xi \delta \theta^k - \delta u_a^k \right] \psi^k dx dt. \end{aligned} \quad (6)$$

After several integrations by parts with respect to x and t , $\psi^k(x, t)$ is solution of the following adjoint problem:

$$\begin{cases} -\rho C \frac{\partial \psi^k(\cdot)}{\partial t} - \lambda(\theta) \frac{\partial^2 \psi^k(\cdot)}{\partial x^2} = \xi \psi^k(\cdot) & \forall (x, t) \in \Omega \times T \\ \psi^k(0, t) = \psi^k(L, t) = 0 & \forall t \in T \\ \psi^k(x, t_f) = -\frac{1}{\rho C} \theta^k(x, t_f) & \forall x \in \Omega \end{cases} \quad (7)$$

This system is numerically solved using a retrograde scheme in time. For $\psi^k(x, t)$ solution of system (7), variation of Lagrangian is:

$$\delta \ell(u_a^k, \theta^k, \psi^k) = \frac{\partial \ell}{\partial u_a^k} \delta u_a^k = \delta J(u_a^k).$$

According to the previous equations, cost function gradient is defined as follows:

$$\frac{\partial J}{\partial u_{a_j}^k} = - \int_0^{t_f} \int_{a-\varepsilon}^{a+\varepsilon} \psi^k(x, t) s_j(t) dx dt \quad (8)$$

The descent direction can be estimated at each new iteration k from the previous gradient (at iteration $k-1$), as follows:

$$\mathbf{d}^k = - \left(\frac{\partial J}{\partial u_{a_j}^k} \right)_{j=1, \dots, M} + \beta_k \mathbf{d}^{k-1} \quad (9)$$

$$\text{with } \beta_k = \frac{\left\| \left(\frac{\partial J}{\partial u_{a_j}^k} \right)_{j=1, \dots, M} \right\|^2}{\left\| \left(\frac{\partial J}{\partial u_{a_j}^{k-1}} \right)_{j=1, \dots, M} \right\|^2} \text{ and } \|\cdot\| \text{ is the Euclidean norm.}$$

3.3 Conjugate gradient algorithm

In order to solve the previous inverse problem, an offline algorithm is implemented as follows:

1. Initialization of the unknown parameter (control flux) at the first iteration $k = 0$: $[u_{aj}^k]_{j=1,\dots,M} = 0$,
2. Resolution of the direct problem (1) in order to determine $\theta^k(x, t)$,
3. Determination of the criterion $J(\theta^k)$ according to (2):
 - If the criterion is below the stopping threshold J_{stop} , then the algorithm is stopped and u_a^k is considered as the estimate of the unknown parameter (internal control law $u_a(x, t)$ is then identified).
 - Otherwise, continue to step 4,
4. Solve the adjoint problem (7) in order to determine the Lagrange multiplier $\psi^k(x, t)$ and the cost function gradient according to (8). Determination of the descent direction \mathbf{d}^k according to (9),
5. Resolution of the sensitivity problem (3) in the descent direction \mathbf{d}^k to calculate the sensitivity function $\delta\theta^k(x, t)$ and determination of the descent depth γ^k according to (5),
6. Update new estimations of the control according to (4),
7. Increment of the iteration $k = k + 1$ and back to step 2.

In the next section, several numerical configurations are investigated in order to highlight the method relevance.

4 Numerical results

In this paragraph, specific cases are proposed considering thermal properties given in section 2.2. Non stability is induced by $\xi = 4 \times 10^4$ and without control system evolution is shown in Fig. 3. The previous numerical method is used for the determination of a relevant control law and the following values presented in Table. 1 are taken into account with $J_{stop} = 10^{-3}$:

Table 1. *The various cases considered.*

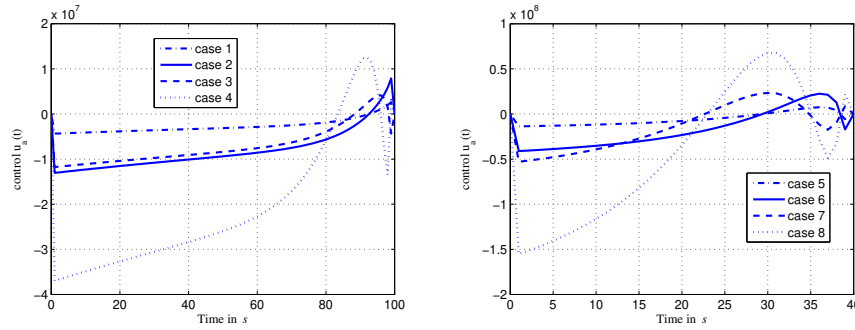
	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
t_f	100	100	100	100	40	40	40	40
$a \times 10^{-2}$	3	3	9	9	3	3	9	9
$\varepsilon \times 10^{-3}$	3	1	3	1	3	1	3	1

Furthermore, in order to compare the previous cases together, the energy required by each control law is calculated as follows: $\mathcal{E} = \int_0^L \int_0^{t_f} [u_a(x, t)]^2 dx dt$ and is given in the following Table. 2.

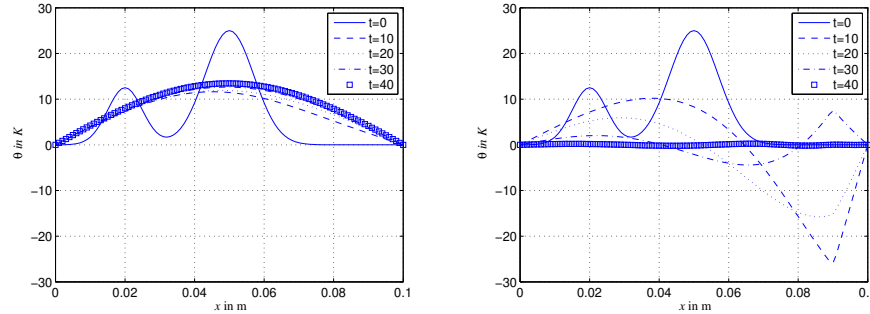
Table 2. *The cost of each cases.*

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$\mathcal{E} \times 10^{12} \simeq$	2.8	8.5	20	63	9.3	28	99	290

In Fig. 4. several results are shown: the evolution of identified control u_{aj} for cases 1, 2, 3 and 4 on the left and the evolution of identified control u_{aj} for cases 5, 6, 7 and 8 on the right.


Fig. 4. Evolution of identified control laws: cases 1, 2, 3 and 4 (left), cases 5, 6, 7 and 8 (right).

In Fig. 5. temperature evolution without control on the left and with control on the right for the case 8 are shown.


Fig. 5. Temperature evolution without control (left) and with control (right) - case 8.

According to the previous results, several remarks could be proposed:

- If the spatial support ω of the actuator is small, then the control flux has to be big. Comparing cases 1&2, 3&4, 5&6 and 7&8, it is shown that if ω is divided by 3, then the energy required by the control is multiplied by 3.
- If the final time t_f of the process is small, then the control flux has to be big. Moreover, due to the small time horizon and to the wide control

flux, evolution of the control law is complex (with positive and negative commutation). This behaviour is meaningful and could be compare to a bang-bang point wise controller.

5 Conclusions

In this paper, the stabilization of a non linear parabolic PDE system in $1D$ geometry has been investigated. Considering a thermal realistic configuration, non stability has been numerically put in evidence. In order to ensure the convergence of the system state toward zero, an in-domain actuator has been implemented. An iterative regularization method for identification of such unknown control is presented. Numerical results of in-domain control determination for several studied configuration are provided and show that the suggested method is relevant in order to identify an in-domain control law in such non linear system. Effect of actuator spatial support (size, location) has been studied and it has been shown how the desired final time (for stabilization purposes) affect the determination of the control.

Future works will consist in adaptation of online implementation in order to ensure disturbances rejection. Moreover, the effect of several non collocated actuators and sensors could be easily investigated considering the proposed method. The control of few moving actuators seems to be quite attractive and could ensure null controllability in an elegant way.

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