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40	Abstract	<p>Timed Event Graphs (TEGs) are a graphical model for decision free and time-invariant Discrete Event Systems (DESS). To express systems with time-variant behaviors, a new form of synchronization, called partial synchronization (PS), has been introduced for TEGs. Unlike exact synchronization, where two transitions t_1, t_2 can only fire if both transitions are simultaneously enabled, PS of transition t_1 by transition t_2 means that t_1 can fire only when transition t_2 fires, but t_1 does not influence the firing of t_2. This, for example can describe the synchronization between a local train and a long distance train. Of course it is reasonable to synchronize the departure of a local train by the arrival of long distance train in order to guarantee a smooth connection for passengers. In contrast, the long distance train should not be delayed due to the late arrival of a local train Under the assumption that PS is periodic, we can show that the dynamic behavior of a TEG under PS can be decomposed into a time-variant and a time-invariant part. It is shown that the time-variant part is invertible and that the time-invariant part can be modeled by a matrix with entries in the dioid $\text{Minax}\gamma, \delta$, i.e. the time-invariant part can be interpreted as a standard TEG. Therefore, the tools introduced for standard TEGs can be used to analyze and to control the overall system. In particular, in this paper output reference control for TEGs under PS is addressed. This control strategy determines the optimal input for a predefined reference output. In this case optimality is in the sense of the "just-in-time" criterion, i.e., the input events are chosen as late as possible under the constraint that the output events do not occur later than required by the reference output.</p>	
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Model decomposition of timed event graphs under periodic partial synchronization: application to output reference control

Johannes Trunk¹ · Bertrand Cottenceau² · Laurent Hardouin² · Joerg Raisch¹

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Abstract

Timed Event Graphs (TEGs) are a graphical model for decision free and time-invariant Discrete Event Systems (DESs). To express systems with time-variant behaviors, a new form of synchronization, called partial synchronization (PS), has been introduced for TEGs. Unlike exact synchronization, where two transitions t_1, t_2 can only fire if both transitions are simultaneously enabled, PS of transition t_1 by transition t_2 means that t_1 can fire only when transition t_2 fires, but t_1 does not influence the firing of t_2 . This, for example can describe the synchronization between a local train and a long distance train. Of course it is reasonable to synchronize the departure of a local train by the arrival of long distance train in order to guarantee a smooth connection for passengers. In contrast, the long distance train should not be delayed due to the late arrival of a local train Under the assumption that PS is periodic, we can show that the dynamic behavior of a TEG under PS can be decomposed into a time-variant and a time-invariant part. It is shown that the time-variant part is invertible and that the time-invariant part can be modeled by a matrix with entries in the dioid $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, i.e. the time-invariant part can be interpreted as a standard TEG. Therefore, the tools introduced for standard TEGs can be used to analyze and to control the overall system. In particular, in this paper output reference control for TEGs under PS is addressed. This control strategy determines the optimal input for a predefined reference output. In this case optimality is in the sense of the "just-in-time" criterion, i.e., the input events are chosen as late as possible under the constraint that the output events do not occur later than required by the reference output.

Keywords Dioids · Optimal control · TEG · Discrete-event systems · Residuation · Time-variant behaviour

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28 1 Introduction and motivation

29 TEGs are a subclass of timed Petri nets where each place has exactly one input and one
30 output transition and all arcs have weight 1. Timed Event Graphs under Partial Synchron-
31 ization (TEGsPS) are an extension of TEGs introduced in David-Henriet et al. (2014). A
32 similar extension was introduced in De Schutter and van den Boom (2003), where TEGs
33 with hard and soft synchronization are studied. TEGsPS can express some time-variant phe-
34 nomena which cannot be expressed by standard TEGs. For instance, partial synchronization
35 (PS) is useful to model systems where particular events can only occur in a specific time
36 window. E.g., at an intersection, a vehicle can only cross when the traffic light is green.
37 Clearly this describes a time-variant behavior, since the vehicle is delayed by a time that
38 depends on its time of arrival at the intersection. If an earliest functioning rule is adopted,
39 the behavior of a TEG can be modeled by linear equations in a specific algebraic structure
40 called dioid. Based on such dioids, a general theory has been developed for performance
41 evaluation and control of TEGs, e.g. Baccelli et al. (1992) and Heidergott et al. (2005). In
42 particular, the problem of output reference control for TEGs was studied in Baccelli et al.
43 (1992); Cohen et al. (1989); Menguy et al. (1998, 2000). Recently, in David-Henriet et al.
44 (2014, 2015, 2016), dioid theory has been applied to TEGsPS and first results have been
45 obtained for performance evaluation and controller synthesis for TEGsPS. In David-Henriet
46 et al. (2014) output reference control was introduced for TEGsPS. There, the earliest evolu-
47 tion of a Timed Event Graph under Partial Synchronization (TEGPS) is modeled as a
48 $(\max, +)$ -system with additional constraints. The control problem is then solved for a finite
49 reference output by solving the backward equation for this $(\max, +)$ -system. In Hamaci et al.
50 (2006) and Trunk et al. (2017b) output reference control was studied for TEGs with positive
51 integer weights on the arcs. These TEGs exhibit event-variant behavior and can therefore be
52 seen as the counter-part to TEGsPS.

53 In this paper we investigate TEGsPS where partial synchronization is periodic. To con-
54 sider only periodic partial synchronization is not overly restrictive as periodic schedules are
55 common in many applications. E.g. in transportation networks: many public transportation
56 system as well as freight railway services work with a periodic schedule. Similarly in manu-
57 facturing systems: there are many production processes, where a resource is shared between
58 several machines on the basis of a periodic schedule. We show that for TEGsPS with peri-
59 odic PS the dynamic behavior can be modeled in a specific dioid called $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$. A specific
60 time-variant operator is introduced to take PS into account. Similar to transfer functions
61 for standard TEGs in the dioid $\mathcal{M}_{in}^{ax}[\llbracket \gamma, \delta \rrbracket]$, the transfer behavior of TEGsPS is described
62 by ultimately cyclic series in the dioid $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$. These transfer functions are useful, for
63 instance, for computing the output for a given input of a system, for system composition
64 and for control synthesis.

65 This paper is organized as follows: Section 2 summarizes the necessary facts on TEGsPS
66 and dioid theory. In Section 3, modeling of TEGsPS in the dioid $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ is introduced.
67 Section 4 discusses a decomposition method for elements in $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ and provides tools to
68 handle operations on ultimately cyclic series in $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$. In particular, we show that basic
69 operations on ultimately cyclic series in $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ can be reduced to operations between
70 matrices in $\mathcal{M}_{in}^{ax}[\llbracket \gamma, \delta \rrbracket]$. In Section 5, transfer functions for TEGsPS in $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ are used
71 to solve the optimal output reference control problem for this system class.

72 A preliminary version of this work has been reported in Trunk et al. (2018), where the
73 modeling process of a TEGPS in the dioid $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ was established and a decomposition
74 into an invertible time-variant and a time-invariant part was discussed. The purpose of this
75 paper is to introduce optimal output reference control for TEGsPS based on the model in

the dioid $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$. As a prerequisite, results on the residuation of the product in the dioid $\mathcal{T}_{per}[\llbracket \gamma \rrbracket]$ are obtained.

2 Timed event graphs and dioids

2.1 Timed event graphs

In the following, we briefly recall the necessary facts on TEGs. For details, see Baccelli et al. (1992) and Heidergott et al. (2005). A TEG consists of a set of places $P = \{p_1, \dots, p_n\}$, a set of transitions $T = \{t_1, \dots, t_m\}$ and a set of arcs $A \subseteq (P \times T) \cup (T \times P)$, all with weight 1. Place p_i is an upstream place of transition t_j (and transition t_j is a downstream transition of place p_i), if $(p_i, t_j) \in A$. Conversely, p_i is a downstream place of transition t_j (and t_j is an upstream transition of place p_i), if $(t_j, p_i) \in A$. For TEGs, each place p_i has exactly one upstream transition and exactly one downstream transition. Moreover, each place p_i exhibits an initial marking $(\mathcal{M}_0)_i \in \mathbb{N}_0$ and a holding time $(\phi)_i \in \mathbb{N}_0$. A transition t_j is said to be enabled, if the marking in every upstream place is at least 1. When t_j fires, the marking $(\mathcal{M})_i$ in every upstream place p_i is reduced by 1 and the marking $(\mathcal{M})_o$ in every downstream place p_o is increased by 1. The holding time $(\phi)_i$ is the time a token must remain in place p_i before it contributes to the firing of the downstream transition of p_i . The set T of transitions is partitioned into input transitions, i.e., transitions without upstream places, output transitions, i.e., transitions without downstream places and internal transitions, i.e., transitions with both upstream and downstream places. We say that a TEG is operating under the earliest functioning rule, if all internal and output transitions are fired as soon as they are enabled.

2.2 Timed event graphs under partial synchronization

TEGsPS provide a suitable model for some time-variant discrete event systems. In the following, we give a brief introduction. For further information the reader is invited to consult (David-Henriet et al. 2014). Considering the TEG in Fig. 1a, assuming the earliest functioning rule, incoming tokens in place p_1 are immediately transferred to place p_2 by the firing of transition t_2 , as the holding time of place p_1 is zero. Note that zero holding times are, by convention, not indicated in visual illustrations of TEGs. In contrast, Fig. 1b illustrates a TEG with PS of transition t_2 by transition t_a . This means that t_2 can only fire if t_a fires, but the firing of t_a does not depend on t_2 .

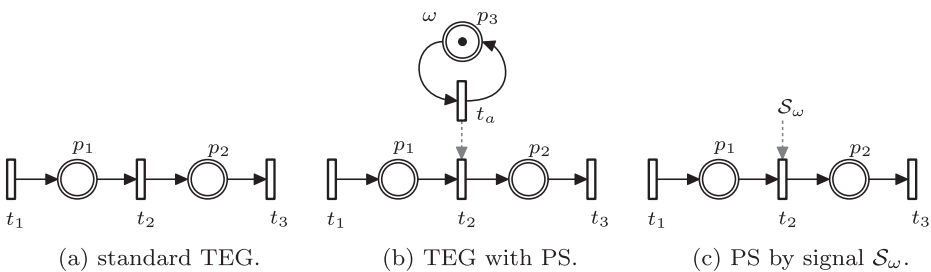


Fig. 1 a standard TEG. b PS of t_2 by t_a , triggered every ω time units. c equivalent PS expressed by a signal S_ω

106 In this example, place p_3 (equipped with a holding time of ω) and transition t_a , together
 107 with the corresponding arcs, constitute an autonomous TEG. Under the earliest functioning
 108 rule, the firings of transition t_a generate a periodic signal \mathcal{S}_ω with a period $\omega \in \mathbb{N}$. Therefore,
 109 the PS of t_2 by t_a can also be described by a predefined signal $\mathcal{S}_\omega: \mathbb{Z} \rightarrow \{0, 1\}$, enabling the
 110 firing of t_2 at times t where $\mathcal{S}_\omega(t) = 1$. In particular, $\mathcal{S}_\omega(t) = 1$ if $t \in \{j\omega \text{ with } j \in \mathbb{Z}\}$ and
 111 0 otherwise.

112 **Definition 1** A periodic signal $\mathcal{S}: \mathbb{Z} \rightarrow \{0, 1\}$ is defined by a string $\langle n_0, n_1, \dots, n_I \rangle$, with
 113 $n_i \in \mathbb{N}_0, 0 \leq i \leq I$ and a period $\omega \in \mathbb{N}$, such that $\forall j \in \mathbb{Z}$

$$\mathcal{S}(t) = \begin{cases} 1 & \text{if } t \in \{n_0 + \omega j, n_1 + \omega j, \dots, n_I + \omega j\}, \\ 0 & \text{otherwise,} \end{cases}$$

114 where the string $\langle n_0, n_1, \dots, n_I \rangle$ is strictly increasing, i.e., $\forall i \in \{1, \dots, I\}, n_{i-1} < n_i$,
 115 and $n_I < \omega$.

116 *Example 1* The signal

$$\mathcal{S}_1(t) = \begin{cases} 1 & \text{if } t \in \{\dots, -4, -3, 0, 1, 4, 5, 8, 9, \dots\}, \\ 0 & \text{otherwise,} \end{cases}$$

117 is a periodic signal with a period $\omega = 4$ and a string $\langle 0, 1 \rangle$. Therefore $\forall j \in \mathbb{Z}$,

$$\mathcal{S}_1(t) = \begin{cases} 1 & \text{if } t \in \{0 + 4j, 1 + 4j\}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

118 **Definition 2** A Timed Event Graph under periodic partial synchronization is a TEG where
 119 the firings of some internal and output transitions are synchronized with periodic signals.

120 Note that the assumption that only internal and output transitions are subject to PS is
 121 not restrictive since we can always add new input transitions and extend the set of internal
 122 transitions by the former input transitions. In David-Henriet et al. (2015), ultimately periodic
 123 signals are considered for PS of transitions. It was shown that the behavior of a TEGPS
 124 with such synchronization signals can be described by recursive equations in state space
 125 form. In this work, we focus on (immediately) periodic signals for PS of transitions. To
 126 consider only periodic PS allows us to define a dioid of operators to describe the behavior
 127 of TEGsPS. In particular, we can show that the transfer behavior of a TEGPS is described
 128 by a rational power series of an ultimately cyclic form. Let us note that focusing on periodic
 129 signals for a PS of a transition is not overly restrictive as periodic schedules are common in
 130 many applications.

131 *Example 2* Let us consider a simple supply chain between two factories. Factory 1 is a
 132 supplier for factory 2. The products of factory 1 are transported via a train connection to
 133 factory 2. This simple supply chain is modelled by the TEG under periodic PS shown in
 134 Fig. 2, with periodic PS of transition t_2 by the signal, $\forall j \in \mathbb{Z}$

$$\mathcal{S}_2(t) = \begin{cases} 1 & \text{if } t \in \{1 + 20j\}, \\ 0 & \text{otherwise.} \end{cases}$$

135 Transition t_1 models the issue of the goods at factory 1 and transition t_4 the receipt
 136 of goods at factory 2. Transition t_2, t_3 and places p_2, p_3 model the train line between the
 137 factories. The holding time of 10 time units of place p_3 models the travel time of trains
 138 between the factories. The 2 initial tokens in place p_2 describe the maximal capacity of the

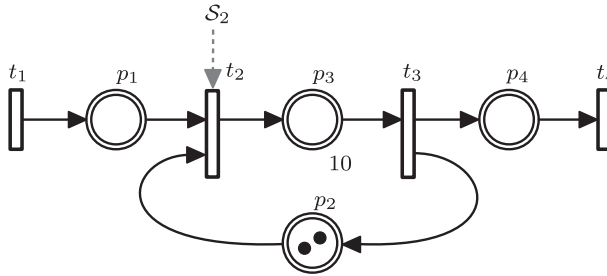


Fig. 2 Example of a TEGPS

trains. The schedule of the trains is modelled by the signal S_2 , hence every 20 time units there is a train leaving from factory 1. We will recall this example again in Section 5 and demonstrate how "just-in-time" control for this supply chain can be computed using the methods developed in this paper.

2.3 Dioids

A dioid \mathcal{D} is an algebraic structure with two binary operations, \oplus (addition) and \otimes (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D}, a \oplus a = a$). The neutral element for addition, denoted by ε , is absorbing for multiplication (i.e., $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element denoted by e . The element e (resp. ε) is called unit (resp. zero) element of the dioid.

Note that, as in conventional algebra, the multiplication symbol \otimes is often omitted. A dioid \mathcal{D} is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid is a partially ordered set, with a canonical order \geq defined by $a \oplus b = a \Leftrightarrow a \geq b$. The infimum operator can then be defined by $a, b \in \mathcal{D}, a \wedge b = \bigoplus\{x \in \mathcal{D} \mid x \oplus a \leq a, x \oplus b \leq b\}$. Moreover, in a complete dioid, the Kleene star of an element $a \in \mathcal{D}$, denoted a^* , is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = e$ and $a^{i+1} = a \otimes a^i$. Let $\mathcal{C} \subseteq \mathcal{D}$ then \mathcal{C} is a subdioid of \mathcal{D} if e and ε are in \mathcal{C} and \mathcal{C} is closed for \oplus and \otimes .

Theorem 1 (Baccelli et al. 1992) *In a complete dioid \mathcal{D} , $x = a^*b$ is the least solution of the implicit equation $x = ax \oplus b$.*

Here, the adjective "least" refers to the canonical order in the dioid described above.

Both multiplication and addition on a (complete) dioid \mathcal{D} can be readily extended to the matrix case: for matrices $A, B \in \mathcal{D}^{m \times n}$, $C \in \mathcal{D}^{n \times q}$ and a scalar $\lambda \in \mathcal{D}$, matrix addition and multiplication are defined by

$$(A \oplus B)_{i,j} := (A)_{i,j} \oplus (B)_{i,j}, \quad (\lambda \otimes A)_{i,j} := \lambda \otimes (A)_{i,j},$$

$$(A \otimes C)_{i,j} := \bigoplus_{k=1}^n ((A)_{i,k} \otimes (C)_{k,j}).$$

Moreover, the order relation on matrices of the same dimension is understood elementwise, i.e. $A \geq B$ iff $(A)_{i,j} \geq (B)_{i,j}, \forall i, j$. The identity matrix, denoted by I , is a square matrix with elements e on the diagonal and ε otherwise.

166 **2.4 Complete dioids and residuation theory**

167 Residuation theory is a formalism to address the problem of approximate mapping inversion
 168 over ordered sets (Baccelli et al. 1992). It applies to complete dioids, since a complete dioid
 169 \mathcal{D} is a partially ordered set.

170 **Definition 3** (Baccelli et al. 1992) A mapping $f : \mathcal{D} \rightarrow \mathcal{L}$, with \mathcal{D} and \mathcal{L} complete dioids,
 171 is residuated if $\forall b \in \mathcal{L}$ the inequality $f(x) \leq b$ has a greatest solution in \mathcal{D} , denoted $f^\sharp(b)$.
 172 The mapping $f^\sharp : \mathcal{L} \rightarrow \mathcal{D}$, is called the residual of f .

173 **Theorem 2** (Baccelli et al. 1992) A mapping $f : \mathcal{D} \rightarrow \mathcal{L}$, with \mathcal{D} and \mathcal{L} complete dioids,
 174 is residuated iff $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous, that is

$$f\left(\bigoplus_{x \in X} x\right) = \bigoplus_{x \in X} f(x),$$

175 for every (finite or infinite) subset X of \mathcal{D} .

176 On a complete dioid the mapping $R_a : x \mapsto xa$, (right multiplication by a) resp.
 177 $L_a : x \mapsto ax$ (left multiplication by a), is lower-semicontinuous and therefore residuated.
 178 The residual mappings are denoted $R_a^\sharp(b) = b \not\phi a = \bigoplus\{x \mid xa \leq b\}$ (right division by a)
 179 and $L_a^\sharp(b) = a \not\backslash b = \bigoplus\{x \mid ax \leq b\}$ (left division by a). Left and right division can be
 180 extended to the matrix case. For matrices $A \in \mathcal{D}^{m \times n}$, $B \in \mathcal{D}^{m \times q}$, $C \in \mathcal{D}^{n \times q}$

$$(A \not\backslash B)_{i,j} = \bigwedge_{k=1}^m ((A)_{k,i} \not\backslash (B)_{k,j}), \quad (B \not\phi C)_{i,j} = \bigwedge_{k=1}^q ((B)_{i,k} \not\phi (C)_{j,k}). \quad (2)$$

181 In the following some useful properties of left and right division are summarized, for a
 182 proof see Baccelli et al. (1992) or the recent summary paper (Hardouin et al. 2018). For
 183 $a, b, x \in \mathcal{D}$ and \mathcal{D} a complete dioid,

$$(ab) \not\backslash x = b \not\backslash (a \not\backslash x) \quad x \not\phi (ba) = (x \not\phi a) \not\phi (b) \quad (3)$$

$$(a \oplus b) \not\backslash x = (a \not\backslash x) \wedge (b \not\backslash x) \quad x \not\phi (a \oplus b) = (x \not\phi a) \wedge (x \not\phi b), \quad (4)$$

$$(a \not\backslash x) \not\phi b = a \not\backslash (x \not\phi b). \quad (5)$$

184 **3 Modeling of TEGs under PS in the Dioid $\mathcal{T}[\mathbb{Y}]$**

185 To model TEGsPS, a dater function $x_i : \mathbb{Z} \rightarrow \mathbb{Z}_{max} := \{\mathbb{Z}\} \cup \{\infty\} \cup \{-\infty\}$ is associated to
 186 each transition t_i . The value $x_i(k)$ gives the date (time) when transition t_i fires the $(k+1)^{st}$
 187 time. Naturally, dater functions are nondecreasing functions, i.e., $x_i(k+1) \geq x_i(k)$. The
 188 set of dater functions is denoted by Σ . On Σ , addition and multiplication by a constant are
 189 defined as follows:

$$x, y \in \Sigma, (x \oplus y)(k) := \max(x(k), y(k)),$$

$$\lambda \in \mathbb{Z}_{max}, (\lambda \otimes x)(k) := \lambda + x(k).$$

190 The zero element $\tilde{\varepsilon}$ on Σ is defined by $\tilde{\varepsilon}(k) = -\infty, \forall k \in \mathbb{Z}$. The \oplus operation induces an
 191 order relation on Σ , i.e., for $x, y \in \Sigma, x \leq y \Leftrightarrow x \oplus y = y$. In this order, the top element
 192 $\tilde{\top}$ is defined by $\tilde{\top}(k) = +\infty, \forall k \in \mathbb{Z}$. An operator, i.e., a map, $o : \Sigma \rightarrow \Sigma$ is linear if (a)
 193 $\forall x, y \in \Sigma : o(x \oplus y) = o(x) \oplus o(y)$ and (b) $\lambda \otimes o(x) = o(\lambda \otimes x)$. An operator is additive if

(a) is satisfied. Let \mathcal{O} denote the set of all operators $o : \Sigma \rightarrow \Sigma$. Moreover, let \mathcal{O}_a denote the subset of all additive operators in \mathcal{O} . 194
195

Proposition 1 (Cottenceau et al. 2014) *The set \mathcal{O}_a equipped with addition and multiplication: $x \in \Sigma, \forall o_1, o_2 \in \mathcal{O}_a$,* 196
197

$$(o_1 \oplus o_2)(x) := o_1(x) \tilde{\oplus} o_2(x), (o_1 \otimes o_2)(x) := o_1(o_2(x)), \tag{6}$$

is a noncommutative complete dioid. The identity operator (unit element) is denoted by $e : \forall x \in \Sigma, e(x) = x$, the zero operator (zero element) is denoted by $\varepsilon : \forall x \in \Sigma, \varepsilon(x) = \tilde{\varepsilon}$ and the top operator (top element) is denoted by $\top : \forall x \in \Sigma \setminus \{\tilde{\varepsilon}\}, \top(x) = \tilde{\top}$. 198
199
200

To simplify notation, we write ox instead of $o(x)$ wherever clear from the context. 201

Definition 4 (Basic operators in \mathcal{O}) Dynamic phenomena arising in TEGsPS can be described by the following basic operators in \mathcal{O} : 202
203

$$\tau \in \mathbb{Z}, \delta^\tau : \forall x \in \Sigma, (\delta^\tau x)(k) = x(k) + \tau, \tag{7}$$

$$\eta \in \mathbb{Z}, \gamma^\eta : \forall x \in \Sigma, (\gamma^\eta x)(k) = x(k - \eta), \tag{8}$$

$$\omega, \varpi \in \mathbb{N}, \Delta_{\omega|\varpi} : \forall x \in \Sigma, (\Delta_{\omega|\varpi} x)(k) = \lceil x(k)/\varpi \rceil \omega, \tag{9}$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a . 204

It can be easily checked that all these operators are additive, i.e., $\delta^\tau, \gamma^\eta, \Delta_{\omega|\varpi} \in \mathcal{O}_a$. The time- and event-shift operator δ and γ are used to model the dynamic behavior of standard TEGs, e.g., Baccelli et al. (1992). In addition we introduce the $\Delta_{\omega|\varpi}$ operator to consider phenomena caused by PS. 205
206
207
208

Proposition 2 (Trunk et al. 2018) *The basic operators satisfy the following relations* 209

$$\gamma^\eta \oplus \gamma^{\eta'} = \gamma^{\min(\eta, \eta')}, \quad \delta^\tau \oplus \delta^{\tau'} = \delta^{\max(\tau, \tau')}, \tag{10}$$

$$\gamma^\eta \otimes \gamma^{\eta'} = \gamma^{\eta + \eta'}, \quad \delta^\tau \otimes \delta^{\tau'} = \delta^{\tau + \tau'}, \tag{11}$$

$$\Delta_{\omega|\varpi} \otimes \delta^\varpi = \delta^\omega \otimes \Delta_{\omega|\varpi}. \tag{12}$$

Remark 1 Equation 12 implies that for $-b < \tau \leq 0, \Delta_{\omega|b} \delta^\tau \Delta_{b|\varpi} = \Delta_{\omega|\varpi}$, since, 210

$$\begin{aligned} (\Delta_{\omega|b} \delta^\tau \Delta_{b|\varpi} x)(k) &= \left\lceil \frac{\lceil x(k)/\varpi \rceil b + \tau}{b} \right\rceil \omega = \left\lceil \left\lceil \frac{x(k)}{\varpi} \right\rceil + \frac{\tau}{b} \right\rceil \omega \\ &= \left\lceil \frac{x(k)}{\varpi} \right\rceil \omega \quad \text{since } -1 < \tau/b \leq 0, \\ &= (\Delta_{\omega|\varpi} x)(k). \end{aligned}$$

3.1 Dioid of time operators \mathcal{T} 211

In the following, we introduce a dioid of specific time operators in order to model the time-variant behavior of periodic PS. 212
213

214 **Definition 5** (Dioïd of T-operators \mathcal{T}) We denote by \mathcal{T} the dioïd of operators obtained by
 215 addition and composition of operators in $(\varepsilon, e, \delta^s, \Delta_{\omega|\varpi}, \top)$ with $s \in \mathbb{Z}$, and $\omega, \varpi \in \mathbb{N}$.
 216 The elements of \mathcal{T} are called T-operators (T is for time).

217 For example, $\delta^3 \Delta_{4|4} \delta^1 \Delta_{3|2} \in \mathcal{T}$. Since a T-operator only describes a time relation in a
 218 system, e.g., a delay, we can associate a function $\mathcal{R}_v : \mathbb{Z}_{max} \rightarrow \mathbb{Z}_{max}$ to a T-operator v .
 219 This function, when evaluated on t , is obtained by replacing $x(k)$ by t in the expression of
 220 $v(x)(k)$. For example, $((\Delta_{3|4} \delta^1 \oplus \delta^2 \Delta_{3|3})x)(k) = \max(\lceil(x(k) + 1)/4\rceil 3, 2 + \lceil x(k)/3\rceil 3)$
 221 and therefore $\mathcal{R}_{\Delta_{3|4} \delta^1 \oplus \delta^2 \Delta_{3|3}}(t) = \max(\lceil(t + 1)/4\rceil 3, 2 + \lceil t/3\rceil 3)$. The interpretation of \mathcal{R}_v
 222 is as follows. Let x_1 , respectively x_2 , be the dater functions associated with transitions t_1 ,
 223 respectively t_2 . If v maps x_1 to x_2 , then \mathcal{R}_v maps the time of the $(k + 1)^{st}$ firing of t_1 into the
 224 time of the $(k + 1)^{st}$ firing of t_2 . \mathcal{R}_v is therefore called the release-time function associated
 225 to the T-operator v . We denote by \mathcal{R} the set of functions \mathcal{R}_v generated by all operators
 226 v in \mathcal{T} . Clearly, there is an isomorphism between the set of T-operators and the set \mathcal{R} .
 227 The order relation over the dioïd \mathcal{T} corresponds to the order induced by the max operation
 228 on \mathcal{R} .

229 For $v_1, v_2 \in \mathcal{T}$,

$$\begin{aligned} v_1 \geq v_2 &\Leftrightarrow v_1 \oplus v_2 = v_1 \Leftrightarrow v_1 x \tilde{\oplus} v_2 x = v_1 x, \quad \forall x \in \Sigma, \\ &\Leftrightarrow \max((v_1 x)(k), (v_2 x)(k)) = (v_1 x)(k), \quad \forall x \in \Sigma, \quad \forall k \in \mathbb{Z}, \\ &\Leftrightarrow \mathcal{R}_{v_1}(t) \geq \mathcal{R}_{v_2}(t), \quad \forall t \in \mathbb{Z}_{max}. \end{aligned} \tag{13}$$

230 **Definition 6** (Periodic T-operators) A T-operator $v \in \mathcal{T}$ is said to be ω -periodic if its
 231 corresponding function \mathcal{R}_v is quasi- ω -periodic, i.e., $\exists \omega \in \mathbb{N}$ such that $\forall t \in \mathbb{Z}_{max}, \mathcal{R}_v(t +$
 232 $\omega) = \omega + \mathcal{R}_v(t)$. The set of ω -periodic T-operators is denoted by \mathcal{T}_ω . Moreover the set of
 233 periodic operators is defined by $\mathcal{T}_{per} = \bigcup_{\omega \in \mathbb{N}} \mathcal{T}_\omega$.

\mathcal{T}_ω and \mathcal{T}_{per} are subdioïds of \mathcal{T} .

234 *Example 3* The operator $\Delta_{4|4}$ is 4-periodic and the operator $\Delta_{3|3} \delta^2$ is 3-periodic as
 235 $\mathcal{R}_{\Delta_{4|4}}(t) = \lceil t/4 \rceil 4$ and $\mathcal{R}_{\Delta_{3|3} \delta^2}(t) = \lceil (t + 2)/3 \rceil 3$. Therefore $\Delta_{4|4} \in \mathcal{T}_4, \Delta_{3|3} \delta^2 \in \mathcal{T}_3$.
 236 Evidently, both operators are also 12-periodic and therefore $\Delta_{4|4}, \Delta_{3|3} \delta^2 \in \mathcal{T}_{12}$.

237 **Proposition 3** (Trunk et al. 2018) *An ω -periodic T-operator $v \in \mathcal{T}_\omega$ has an ω -periodic*
 238 *canonical form given by a finite sum $\bigoplus_{i=1}^I \delta^{\tau_i} \Delta_{\omega|\omega} \delta^{\tau'_i}$, where $\tau_i < \tau_{i+1} \forall i \in \{1, \dots, I-1\}$,*
 239 *$I \leq \omega$ and $-\omega < \tau'_i \leq 0, \forall i \in \{1, \dots, I\}$.*

240 *Remark 2* Clearly each ω -periodic operator $v \in \mathcal{T}_{per}$ is also $n\omega$ -periodic with $n \geq 1$ and
 241 can be represented as $\bigoplus_{i=1}^I \delta^{\tau_i} \Delta_{n\omega|n\omega} \delta^{\tau'_i}$ and $I \leq n\omega$.

242 **Proposition 4** *The 1-periodic identity operator $e = \Delta_{1|1}$ can be represented in the specific*
 243 *form,*

$$e = \bigoplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega|\omega} \delta^{1+i-\omega}. \tag{14}$$

Proof Recall the isomorphism between T-operators and the set \mathcal{R} . Hence it is sufficient to show that $\mathcal{R}_e = \mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}$. Moreover, since $\forall t \in \mathbb{Z}_{max}, \mathcal{R}_e(t) = t$, it remains to show that $\forall t \in \mathbb{Z}_{max}, \mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(t) = t$.

$$\mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(t) = \max \left(\left\lceil \frac{t+1-\omega}{\omega} \right\rceil \omega, \left\lceil \frac{t+2-\omega}{\omega} \right\rceil \omega - 1, \dots, \left\lceil \frac{t}{\omega} \right\rceil \omega - (\omega - 1) \right). \tag{15}$$

Because $\mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(t)$ is a quasi ω -periodic function, Definition 6, it is sufficient to evaluate Eq. 15 for $t = \{1 - \omega, \dots, 0\}$. This leads to,

$$\begin{aligned} \mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(0) &= \max \left(\left\lceil \frac{1-\omega}{\omega} \right\rceil \omega, \left\lceil \frac{2-\omega}{\omega} \right\rceil \omega - 1, \dots, \left\lceil \frac{0}{\omega} \right\rceil \omega - (\omega - 1) \right) \\ &= 0 \\ \mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(-1) &= \max \left(\left\lceil \frac{-\omega}{\omega} \right\rceil \omega, \left\lceil \frac{1-\omega}{\omega} \right\rceil \omega - 1, \dots, \left\lceil \frac{-1}{\omega} \right\rceil \omega - (\omega - 1) \right) \\ &= -1 \\ \mathcal{R}_{\oplus_{i=0}^{\omega-1} \delta^{-i} \Delta_{\omega} \delta^{1+i-\omega}}(1 - \omega) &= \max \left(\left\lceil \frac{2-2\omega}{\omega} \right\rceil \omega, \left\lceil \frac{3-2\omega}{\omega} \right\rceil \omega - 1, \dots, \left\lceil \frac{1-\omega}{\omega} \right\rceil \omega - (\omega - 1) \right) \\ &= 1 - \omega. \end{aligned}$$

□ 249

Example 4 The identity operator $e = \Delta_{1|1}$ can be represented as $e = \Delta_{3|3} \delta^{-2} \oplus \delta^{-1} \Delta_{3|3} \delta^{-1} \oplus \delta^{-2} \Delta_{3|3}$. Figure 3 illustrates that indeed $\mathcal{R}_e(t) = \max(\mathcal{R}_{\Delta_{3|3} \delta^{-2}}(t), \mathcal{R}_{\delta^{-1} \Delta_{3|3} \delta^{-1}}(t), \mathcal{R}_{\delta^{-2} \Delta_{3|3}}(t))$.

The time-variant behavior caused by a periodic PS of a transition can be conveniently modeled in the dioid \mathcal{T} .

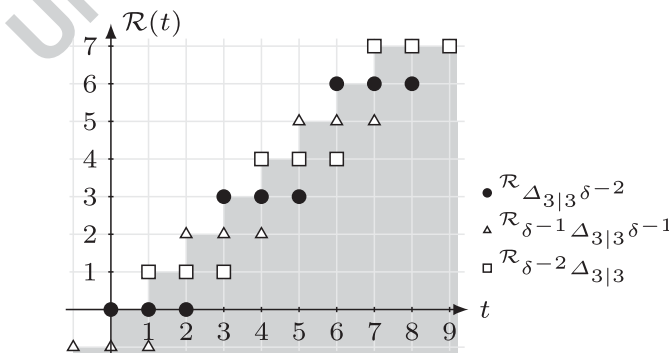


Fig. 3 $\mathcal{R}_e(t) = \max(\mathcal{R}_{\Delta_{3|3} \delta^{-2}}(t), \mathcal{R}_{\delta^{-1} \Delta_{3|3} \delta^{-1}}(t), \mathcal{R}_{\delta^{-2} \Delta_{3|3}}(t))$

255 For this, recall the definition of a periodic signal \mathcal{S} (Definition 1). We associate with
 256 a periodic signal $\mathcal{S} : \mathbb{Z} \rightarrow \{0, 1\}$ characterized by $\langle n_0, \dots, n_I \rangle$ and period ω a function
 257 $\mathcal{R}_S : \mathbb{Z}_{max} \rightarrow \mathbb{Z}_{max}$. This function $\mathcal{R}_S(t)$ is defined by, $\forall j \in \mathbb{Z}$,

$$\mathcal{R}_S(t) = \begin{cases} -\infty & \text{if } t = -\infty \\ n_0 + \omega j & \text{if } (n_I - \omega) + \omega j < t \leq n_0 + \omega j, \\ n_1 + \omega j & \text{if } n_0 + \omega j < t \leq n_1 + \omega j, \\ \vdots & \\ n_I + \omega j & \text{if } n_{I-1} + \omega j < t \leq n_I + \omega j, \\ \infty & \text{if } t = \infty. \end{cases} \quad (16)$$

258 *Example 5* The function $\mathcal{R}_{S_1}(t)$ (Fig. 4b) associated to the signal S_1 (Fig. 4) given in
 259 Example 1 is

$$\mathcal{R}_{S_1}(t) = \begin{cases} -\infty & \text{if } t = -\infty \\ 0 + 4j & \text{if } -3 + 4j < t \leq 0 + 4j, \\ 1 + 4j & \text{if } 0 + 4j < t \leq 1 + 4j, \\ \infty & \text{if } t = \infty. \end{cases}$$

260 The value of $\mathcal{R}_S(t)$ can be interpreted as the next time when the signal \mathcal{S} enables the fir-
 261 ing of the corresponding transition. Clearly, an ω -periodic signal \mathcal{S} leads to a corresponding
 262 function $\mathcal{R}_S(t)$ which satisfies $\forall t \in \mathbb{Z}_{max}, \mathcal{R}_S(t + \omega) = \omega + \mathcal{R}_S(t)$.

263 To prove that a periodic PS of a transition (i.e., the PS is specified by a periodic signal \mathcal{S})
 264 admits an operator representation in the dioid \mathcal{T} , we must show the existence of an operator
 265 $v \in \mathcal{T}$ such that $\mathcal{R}_v = \mathcal{R}_S$.

266 **Proposition 5** (Trunk et al. 2018) *A periodic partial synchronization of a transition by the*
 267 *signal \mathcal{S} in Definition 1 has an operator representation given by*

$$v = \delta^{n_0} \Delta_{\omega|\omega} \delta^{-n_I} \oplus \delta^{n_1 - \omega} \Delta_{\omega|\omega} \delta^{-n_0} \oplus \dots \oplus \delta^{n_I - \omega} \Delta_{\omega|\omega} \delta^{-n_I} \oplus \dots \quad (17)$$

268 *Example 6* Consider the TEGPS shown in Fig. 5, where the signal S_1 is given in Eq. 1
 269 (Example 1) and dater function $x_1(k)$ (resp. $x_2(k)$) is associated with transition t_1 (resp. t_2).

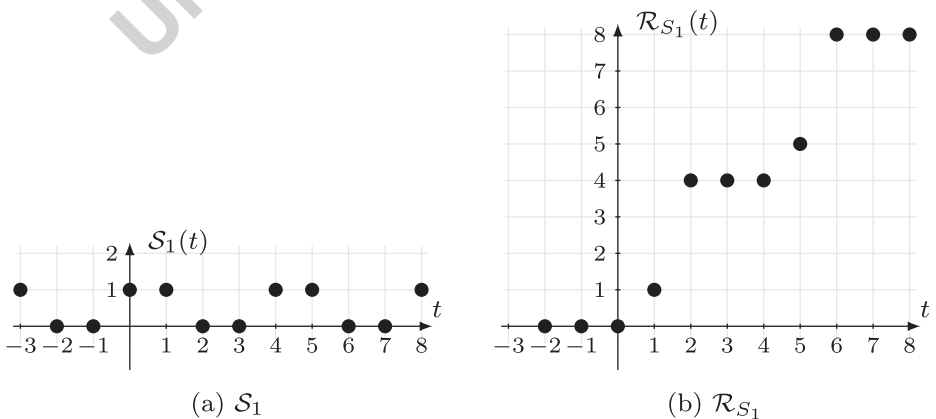
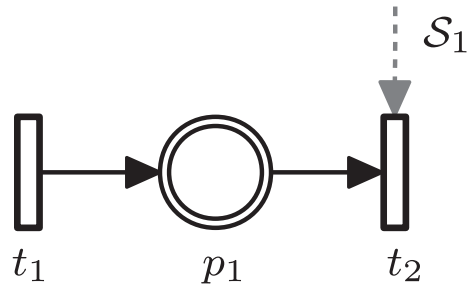


Fig. 4 Signal S_1 and the associated function \mathcal{R}_{S_1}

Fig. 5 Simple TEGPS with a periodic PS of t_2



According to Proposition 5, the behavior of the periodic PS of transition t_2 is modeled by the following operator: 270
271

$$v_{S_1} = \delta^0 \Delta_{4|4} \delta^{-1} \oplus \delta^{-3} \Delta_{4|4} \delta^{-0} = \delta^{-3} \Delta_{4|4} \oplus \Delta_{4|4} \delta^{-1},$$

where the latter equality holds as $\delta^0 = e$. 272

Since the holding time of place p_1 is 0 and there are no initial tokens in the place p_1 this operator describes the firing relation between t_1 and t_2 , i.e., $x_2 = (\delta^{-3} \Delta_{4|4} \oplus \Delta_{4|4} \delta^{-1})x_1$. Therefore, $x_2(k) = \max(-3 + \lceil x_1(k)/4 \rceil, \lceil (x_1(k) - 1)/4 \rceil)$. 273
274
275

Remark 3 Due to the influence of the PS, this firing relation between t_1 and t_2 is time-variant. For instance, if the $(k+1)^{st}$ firing of t_1 is at time instant $x_1(k) = 1$, then the $(k+1)^{st}$ firing of t_2 is at $x_2(k) = 1$, i.e., we have no delay. In contrast, if the $(k+1)^{st}$ firing of t_1 is at time instant $x_1(k) = 2$, then the $(k+1)^{st}$ firing of t_2 is at $x_2(k) = 4$, and the delay is 2. 276
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3.2 Dioid $\mathcal{T}[\gamma]$ 280

Since the γ operator commutes with all T-operators, i.e., $\forall v \in \mathcal{T}, v\gamma = \gamma v$, we can define the dioid $\mathcal{T}[\gamma]$ as follows. 281
282

Definition 7 (Dioid $\mathcal{T}[\gamma]$) We denote by $\mathcal{T}[\gamma]$ the quotient dioid in the set of formal power series in one variable γ with exponents in \mathbb{Z} and coefficients in the noncommutative complete dioid \mathcal{T} induced by the equivalence relation, $\forall s \in \mathcal{T}$, 283
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285

$$s = s(\gamma^*). \tag{18}$$

Hence we identify two series s_1, s_2 with the same equivalence class, if $s_1\gamma^* = s_2\gamma^*$. It is helpful to think of $s\gamma^*$ as the representative of the equivalence class of s . Note that we can interpret elements in $\mathcal{T}[\gamma]$ as nondecreasing functions $s : \mathbb{Z} \rightarrow \mathcal{T}$, where $s(\eta)$ refers to the coefficient of γ^η . Hence, $\forall \eta \in \mathbb{Z}, s(\eta) \leq s(\eta + 1)$. For a fundamental mathematical background on quotient dioids, the reader is invited to consult (Baccelli et al. 1992). Moreover, in Hardouin et al. (2018) quotient dioids are studied from a didactic point of view. 286
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Definition 8 Let $s_1, s_2 \in \mathcal{T}[\gamma]$, then addition and multiplication are defined by 292

$$s_1 \oplus s_2 := \bigoplus_{\eta \in \mathbb{Z}} (s_1(\eta) \oplus s_2(\eta)) \gamma^\eta,$$

$$s_1 \otimes s_2 := \bigoplus_{\eta \in \mathbb{Z}} \left(\bigoplus_{n+n'=\eta} (s_1(n) \otimes s_2(n')) \right) \gamma^\eta.$$

293 We denote by $\mathcal{T}_{per}[\gamma]$ the subdioid of $\mathcal{T}[\gamma]$, obtained by restricting the coefficients v
 294 to periodic operators, i.e., $v \in \mathcal{T}_{per}$. As before, \oplus defines an order on $\mathcal{T}[\gamma]$, i.e., $a, b \in$
 295 $\mathcal{T}[\gamma] : a \oplus b = b \Leftrightarrow a \leq b$. Hence $\forall s_1, s_2 \in \mathcal{T}[\gamma], s_1 \leq s_2 \Leftrightarrow s_1(\eta) \leq s_2(\eta), \eta \in \mathbb{Z}$.
 296 A monomial in $\mathcal{T}[\gamma]$ is defined by $v\gamma^\eta$, where $v \in \mathcal{T}$ and $\eta \in \mathbb{Z}$. The ordering of two
 297 monomials $v_1\gamma^{\eta_1}, v_2\gamma^{\eta_2} \in \mathcal{T}[\gamma]$ can be checked as follows,

$$v_1\gamma^{\eta_1} \leq v_2\gamma^{\eta_2} \Leftrightarrow \begin{cases} v_1 \leq v_2, \\ \eta_1 \geq \eta_2. \end{cases} \tag{19}$$

298 A polynomial is a finite sum of monomials, i.e., $\bigoplus_{i=1}^I v_i\gamma^{\eta_i}$.

299 **Proposition 6** Let $p \in \mathcal{T}_{per}[\gamma]$ be a polynomial, then p has a canonical form $p =$
 300 $\bigoplus_{j=1}^J v'_j\gamma^{\eta'_j}$ such that $\forall j \in \{1, \dots, J\}$, the ω -periodic T -operator v'_j is in the canon-
 301 ical form of Proposition 3, and coefficients and exponents are strictly ordered, i.e., for
 302 $j \in \{1, \dots, J-1\}$, $\eta'_j < \eta'_{j+1}$ and $v'_j < v'_{j+1}$.

303 *Proof* Without loss of generality we can assume that $p = \bigoplus_{i=1}^I v_i\gamma^{\eta_i}$, with $\eta_i < \eta_{i+1}, i =$
 304 $1, \dots, I-1$. In $\mathcal{T}_{per}[\gamma]$, we identify all elements s with $s\gamma^*$, hence can also identify p and

$$p' = \bigoplus_{i=1}^I \left(\bigoplus_{j=1}^i v_j \right) \gamma^{\eta_i}$$

305 as $p\gamma^* = p'\gamma^*$. Hence, $v'_i \leq v'_{i+1}$. If $v'_i = v'_{i+1}$ we can write $v'_i\gamma^{\eta_i} \oplus v'_{i+1}\gamma^{\eta_{i+1}} =$
 306 $v'_i(\gamma^{\eta_i} \oplus \gamma^{\eta_{i+1}}) = v'_i\gamma^{\eta_i}$. Therefore, we can write p' as $\bigoplus_{j=1}^J v'_j\gamma^{\eta'_j}$ with $v'_j < v'_{j+1}$ and
 307 $J \leq I$. □

308 **Definition 9** (*Ultimately Cyclic Series in $\mathcal{T}_{per}[\gamma]$*): A series $s \in \mathcal{T}_{per}[\gamma]$ is said to be
 309 ultimately cyclic if it can be written as $s = p \oplus q(\gamma^\eta\delta^\tau)^*$, where $\eta, \tau \in \mathbb{N}$ and p, q are
 310 polynomials in $\mathcal{T}_{per}[\gamma]$.

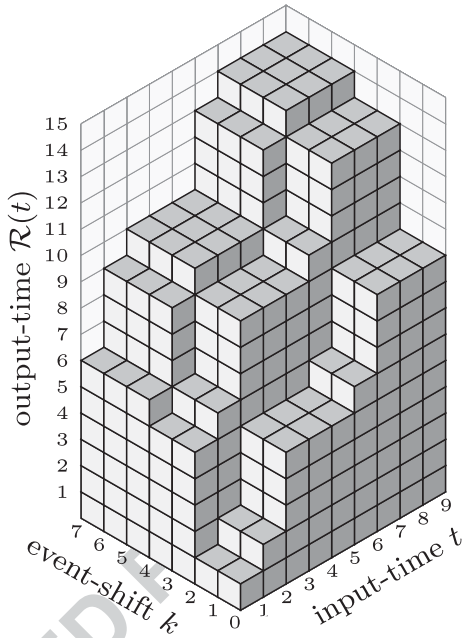
311 An element $s \in \mathcal{T}[\gamma]$ has a three dimensional graphical representation in $\mathbb{Z}_{max} \times \mathbb{Z}_{max} \times$
 312 \mathbb{Z} . Given a series $s = \bigoplus_i v_i\gamma^i \in \mathcal{T}[\gamma]$, this graphical representation is obtained by depict-
 313 ing for every i the release-time function $\mathcal{R}_{v_i} : \mathbb{Z}_{max} \rightarrow \mathbb{Z}_{max}$ of the coefficient v_i in the
 314 (input-time \times output-time)-plane of i .

315 *Example 7* For the graphical representation of the polynomial $p = (\delta^1\Delta_{4|4}\delta^{-1} \oplus$
 316 $\delta^{-2}\Delta_{4|4})\gamma^0 \oplus (\delta^5\Delta_{4|4}\delta^{-1} \oplus \delta^2\Delta_{4|4})\gamma^2 \oplus (\delta^5\Delta_{4|4} \oplus \delta^6\Delta_{4|4}\delta^{-1})\gamma^4 \in \mathcal{T}_{per}[\gamma]$, respectively
 317 its representative

$$\begin{aligned} p\gamma^* &= (\delta^1\Delta_{4|4}\delta^{-1} \oplus \delta^{-2}\Delta_{4|4})\gamma^0 \oplus (\delta^1\Delta_{4|4}\delta^{-1} \oplus \delta^{-2}\Delta_{4|4})\gamma^1, \\ &(\delta^5\Delta_{4|4}\delta^{-1} \oplus \delta^2\Delta_{4|4})\gamma^2 \oplus (\delta^5\Delta_{4|4}\delta^{-1} \oplus \delta^2\Delta_{4|4})\gamma^3 \\ &\oplus (\delta^5\Delta_{4|4} \oplus \delta^6\Delta_{4|4}\delta^{-1})\gamma^4 \oplus (\delta^5\Delta_{4|4} \oplus \delta^6\Delta_{4|4}\delta^{-1})\gamma^5 \oplus \dots, \end{aligned}$$

318 see Fig. 6. The slices in the (I/O-time)-plane for the event-shift values $k = 0, 1$ are illustrated
 319 in Fig. 7a. These slices correspond to the release-time function $\mathcal{R}_{\delta^1\Delta_{4|4}\delta^{-1} \oplus \delta^{-2}\Delta_{4|4}}$ of the
 320 coefficient $\delta^1\Delta_{4|4}\delta^{-1} \oplus \delta^{-2}\Delta_{4|4}$ for γ^0 (resp. γ^1) in p . The slices for $k = 2, 3$ and $k \geq 4$

Fig. 6 3D representation of polynomial
 $p = (\delta^1 \Delta_{4|4} \delta^{-1} \oplus \delta^{-2} \Delta_{4|4}) \gamma^0 \oplus (\delta^5 \Delta_{4|4} \delta^{-1} \oplus \delta^2 \Delta_{4|4}) \gamma^2 \oplus (\delta^5 \Delta_{4|4} \oplus \delta^6 \Delta_{4|4} \delta^{-1}) \gamma^4$.



are shown in Fig. 7b and c. To improve readability, the graphical representation for elements $s \in \mathcal{T}[\gamma]$ has been truncated to non-negative values in Figs. 6 and 7. 321
322

An important subdioid of $\mathcal{T}[\gamma]$ is the dioid $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. This dioid is obtained by restricting the coefficients v to the set $\{\varepsilon, \delta^\tau\}$ of T-operators, i.e., an element in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ is written as $\bigoplus_i \delta^{\tau_i} \gamma^{n_i}$ with $\tau_i, n_i \in \mathbb{Z}$. This dioid has been extensively studied, e.g. Gaubert and Klimann (1991) and Baccelli et al. (1992). The product of two monomials 323
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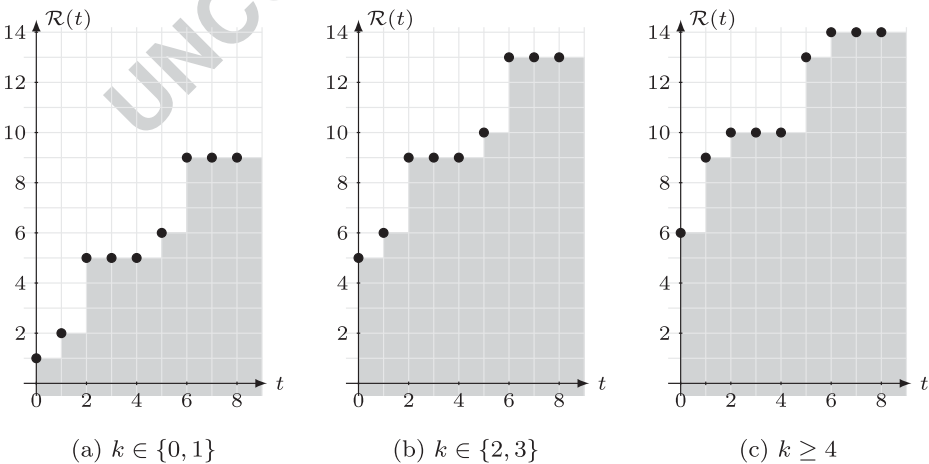


Fig. 7 Slices of the coefficients of p in the (I/O-time)-plane. **a** $\mathcal{R}_{\delta^1 \Delta_{4|4} \delta^{-1} \oplus \delta^{-2} \Delta_{4|4}}$, **b** $\mathcal{R}_{\delta^5 \Delta_{4|4} \delta^{-1} \oplus \delta^2 \Delta_{4|4}}$ and **c** $\mathcal{R}_{\delta^5 \Delta_{4|4} \oplus \delta^6 \Delta_{4|4} \delta^{-1}}$

327 $\gamma^{n_1} \delta^{t_1}, \gamma^{n_2} \delta^{t_2} \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ is obtained by, $\gamma^{n_1} \delta^{t_1} \otimes \gamma^{n_2} \delta^{t_2} = \gamma^{n_1+n_2} \delta^{t_1+t_2}$. Moreover,
 328 Eq. 19 is simplified to $\gamma^{n_1} \delta^{t_1} \leq \gamma^{n_2} \delta^{t_2} \Leftrightarrow (n_1 \geq n_2 \text{ and } t_1 \leq t_2)$, and as a consequence of
 329 Eq. 10,

$$\gamma^n \delta^{t_1} \oplus \gamma^n \delta^{t_2} = \gamma^n \delta^{\max(t_1, t_2)}, \quad \gamma^{n_1} \delta^{t_1} \oplus \gamma^{n_2} \delta^{t_2} = \gamma^{\min(n_1, n_2)} \delta^t. \quad (20)$$

330 A comprehensive description of calculations with series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ can be found
 331 in Baccelli et al. (1992). It is well known that the input-output behavior of a standard
 332 TEG can be described by a transfer function matrix composed of ultimately cyclic series in
 333 $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. Moreover, based on $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, methods for performance evaluation and
 334 controller synthesis have been introduced for TEGs, e.g. Gaubert and Klimann (1991), Maia
 335 et al. (2003), and Hardouin et al. (2017). In (Hardouin et al. 2009), software tools have been
 336 made available for computations in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. The dioid $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ plays a key role
 337 in this paper. In particular, in Section 4, we show that all relevant operations on ultimately
 338 cyclic series $s \in \mathcal{T}_{per} [[\gamma]]$ can be reduced to operations on matrices in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. We can
 339 therefore use the existing tools for $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ to study TEGs under periodic PS.

340 3.3 Modeling of TEGPS in $\mathcal{T}_{per} [[\gamma]]$

341 A TEG under periodic PS operating under the earliest functioning rule admits a representa-
 342 tion in $\mathcal{T}_{per} [[\gamma]]$ of the form

$$\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}. \quad (21)$$

343 This is reminiscent of the state space form in "classical" systems theory. In the sequel, we
 344 will therefore refer to this representation as a state space model. \mathbf{x} (resp. \mathbf{u}, \mathbf{y}) refers to
 345 the vector of dater functions of internal (resp. input, output) transitions. The matrices $\mathbf{A} \in$
 346 $\mathcal{T}_{per} [[\gamma]]^{n \times n}$, $\mathbf{B} \in \mathcal{T}_{per} [[\gamma]]^{n \times g}$ and $\mathbf{C} \in \mathcal{T}_{per} [[\gamma]]^{p \times n}$ describe the influence of transitions
 347 on each other, encoded by operators in $\mathcal{T}_{per} [[\gamma]]$. Hence, n refers to the number of internal
 348 transitions of the TEGPS, while p and q are the number of output and input transitions.
 349 Let us consider a path constituted by the arcs (t_j, p_i) and (p_i, t_o) with a synchronization of
 350 transition t_o by a periodic signal \mathcal{S}_o . The influence of transition t_j on transition t_o is coded
 351 as an operator

$$v_{t_o} \delta^{(\phi)_i} \gamma^{(\mathcal{M}_0)_i}$$

352 where v_{t_o} is the operator representation of the signal \mathcal{S}_o corresponding to the PS of t_o (see
 353 Example 6), $(\phi)_i$ is the holding time of place p_i and $(\mathcal{M}_0)_i$ is the initial marking of p_i .

354 *Example 8* Recall the TEGPS in Fig. 2 with PS of transition t_2 by the signal, $\forall j \in \mathbb{Z}$

$$\mathcal{S}_2(t) = \begin{cases} 1 & \text{if } t \in \{1 + 20j\}, \\ 0 & \text{otherwise.} \end{cases}$$

355 As $\omega = 20, I = 0, n_0 = 1$, according to Proposition 5, $v_{S_2} = \delta^1 \Delta_{20|20} \delta^{-1}$. The influence of
 356 t_3 on transition t_2 via the path $(t_3, p_2)(p_2, t_2)$, is coded by the operator $v_{S_2} \delta^0 \gamma^2 = v_{S_2} \gamma^2 =$
 357 $\delta^1 \Delta_{20|20} \delta^{-1} \gamma^2$. Moreover, by assigning a dater function u (resp. x_1, x_2, y) to transition
 358 t_1 (resp. t_2, t_3, t_4), the earliest functioning of the TEGPS is described in state space form
 359 $\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{B}\mathbf{u}; \mathbf{y} = \mathbf{C}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} \varepsilon & \delta^1 \Delta_{20|20} \delta^{-1} \gamma^2 \\ \delta^{10} & \varepsilon \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \delta^1 \Delta_{20|20} \delta^{-1} \\ \varepsilon \end{bmatrix}, \quad \mathbf{C} = [\varepsilon \ e].$$

360 According to Theorem 1, the least solution of equation $\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{B}\mathbf{u}$ is $\mathbf{x} = \mathbf{A}^* \mathbf{B}\mathbf{u}$.
 361 Therefore, the transfer function matrix \mathbf{H} of a TEGPS can be obtained by $\mathbf{y} = \mathbf{H}\mathbf{u} =$

CA*Bu. In Trunk et al. (2018) it was shown that the entries of the transfer function matrix are ultimately cyclic series in $\mathcal{T}_{per}[\gamma]$. In order to compute this transfer function matrix, to compute system compositions, and to obtain control, we have to perform addition, multiplication and the Kleene star operation of series in $\mathcal{T}_{per}[\gamma]$. In the next section, we show how these operations between series in $\mathcal{T}_{per}[\gamma]$ can be reduced to operations between matrices in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$.

4 Core representation of a series in $\mathcal{T}_{per}[\gamma]$

In this section, we propose a specific decomposition of ultimately cyclic series in $\mathcal{T}_{per}[\gamma]$. We show that such series $s \in \mathcal{T}_{per}[\gamma]$ with period ω can always be represented as $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$ where \mathbf{Q} is a square matrix in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ of size $\omega \times \omega$, \mathbf{m}_ω is a row vector defined as

$$\mathbf{m}_\omega := [\Delta_{\omega|1} \quad \delta^{-1} \Delta_{\omega|1} \quad \dots \quad \delta^{1-\omega} \Delta_{\omega|1}] \tag{22}$$

and \mathbf{b}_ω is a column vector defined as

$$\mathbf{b}_\omega := [\Delta_{1|\omega} \delta^{1-\omega} \quad \dots \quad \Delta_{1|\omega} \delta^{-1} \quad \Delta_{1|\omega}]^T. \tag{23}$$

This representation is called core representation with core matrix \mathbf{Q} . We first demonstrate how to obtain this form on a small example and then provide a formal proof.

Example 9 Consider the following series in $\mathcal{T}_{per}[\gamma]$,

$$s = \Delta_{2|2} \oplus \delta^1 \Delta_{2|2} \delta^{-1} \oplus \delta^2 \Delta_{2|2} \gamma^2 (\delta^2 \gamma^2)^*.$$

Because of $\Delta_{\omega|\omega} = \Delta_{\omega|b} \Delta_{b|\omega}$ (Remark 1) and $\delta^\omega \Delta_{\omega|\omega} = \Delta_{\omega|\omega} \delta^\omega$, see Eq. 12, and as γ commutes with all T-operators, this series can be rewritten as

$$s = \Delta_{2|1} \underbrace{\mathbf{e}}_{M_1} \Delta_{1|2} \oplus \delta^{-1} \Delta_{2|1} \underbrace{\delta^1}_{M_2} \Delta_{1|2} \delta^{-1} \oplus \Delta_{2|1} \underbrace{\delta^1 \gamma^2 (\delta^1 \gamma^2)^*}_{S_1} \Delta_{1|2}.$$

Clearly M_1, M_2 and S_1 are elements in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. We now can rewrite s in the core representation,

$$s = \underbrace{[\Delta_{2|1} \quad \delta^{-1} \Delta_{2|1}]}_{\mathbf{m}_2} \underbrace{\begin{bmatrix} \mathbf{e} & \mathbf{e} \oplus \delta^1 \gamma^2 (\delta^1 \gamma^2)^* \\ \delta^1 & \mathbf{e} \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \Delta_{1|2} \delta^{-1} \\ \Delta_{1|2} \end{bmatrix}}_{\mathbf{b}_2},$$

which is in the required form.

Proposition 7 Let $s = \bigoplus_i v_i \gamma^i \in \mathcal{T}_{per}[\gamma]$ be an ω -periodic series, then s can be written as $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$, where $\mathbf{Q} \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]^{\omega \times \omega}$ and $\mathbf{m}_\omega, \mathbf{b}_\omega$ have the form Eqs. 22 and 23.

Proof s being an ω -periodic series implies that all coefficients v_i of s are ω -periodic T-operators. Then, due to Proposition 3, all coefficients can be expressed in canonical form $v_i = \bigoplus_{j=1}^{J_i} \delta^{\tau_{ij}} \Delta_{\omega|\omega} \delta^{\tau'_{ij}}$ with $J_i \leq \omega$ and $-\omega < \tau'_{ij} \leq 0$. Then s can be rewritten as

$$s = \bigoplus_i \left(\bigoplus_{j=1}^{J_i} \delta^{\tau_{ij}} \Delta_{\omega|\omega} \delta^{\tau'_{ij}} \right) \gamma^i.$$

387 By using $\Delta_{\omega|\omega} = \Delta_{\omega|1} \Delta_{1|\omega}$ (Remark 1), $\delta^\omega \Delta_{\omega|1} = \Delta_{\omega|1} \delta^1$ Eq. 12 and $v\gamma = \gamma v, \forall v \in \mathcal{T}$,
 388 the series s is written as

$$s = \bigoplus_i \left(\bigoplus_{j=1}^{J_i} \delta^{\tilde{\tau}_{ij}} \Delta_{\omega|1} \delta^{\hat{\tau}_{ij}} \gamma^i \Delta_{1|\omega} \delta^{\tau'_{ij}} \right),$$

389 where $-\omega < \tilde{\tau}_{ij} = \tau_{ij} - \lceil \tau_{ij} / \omega \rceil \omega \leq 0$ and $\hat{\tau}_{ij} = \lceil \tau_{ij} / \omega \rceil$. Observe that $-\omega < \tilde{\tau}_{ij}, \tau'_{ij} \leq 0$
 390 hence we can express s by

$$s = \left[\Delta_{\omega|1} \delta^{-1} \Delta_{\omega|1} \cdots \delta^{1-\omega} \Delta_{\omega|1} \right] \left(\bigoplus_i \left(\bigoplus_{j=1}^{J_i} \mathbf{Q}_{ij} \right) \right) \begin{bmatrix} \Delta_{1|\omega} \delta^{1-\omega} \\ \cdots \\ \Delta_{1|\omega} \delta^{-1} \\ \Delta_{1|\omega} \end{bmatrix},$$

391 where the entry $(\mathbf{Q}_{ij})_{1-\tilde{\tau}_{ij}, \omega+\tau'_{ij}} = \delta^{\hat{\tau}_{ij}} \gamma^i$ and all other entries of \mathbf{Q}_{ij} are equal to ε . Hence,
 392 s is in the required form $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$, where $\mathbf{Q} = \bigoplus_i \left(\bigoplus_{j=1}^{J_i} \mathbf{Q}_{ij} \right)$. □

393 Let us note that the core \mathbf{Q} of a series $s \in \mathcal{T}_{per} \llbracket \gamma \rrbracket$ is not unique. In other words, we can
 394 express the same series with different cores, i.e., we may have $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega = \mathbf{m}_\omega \tilde{\mathbf{Q}} \mathbf{b}_\omega$ with
 395 $\mathbf{Q}, \tilde{\mathbf{Q}} \in \mathcal{M}_{in}^{ax} \llbracket \gamma, \delta \rrbracket^{\omega \times \omega}$ but $\mathbf{Q} \neq \tilde{\mathbf{Q}}$. We illustrate this in the following example.

396 *Example 10* Recall the series $s = \Delta_{2|2} \oplus \delta^1 \Delta_{2|2} \delta^{-1} \oplus \delta^2 \gamma^2 (\delta^2 \gamma^2)^* \Delta_{2|2}$ given in Example
 397 9. The series s can be expressed by $\mathbf{m}_2 \tilde{\mathbf{Q}} \mathbf{b}_2$ where,

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \varepsilon & \varepsilon \oplus \delta^1 \gamma^2 (\delta^1 \gamma^2)^* \\ \delta^1 & \varepsilon \end{bmatrix}.$$

398 Clearly $\tilde{\mathbf{Q}} \neq \mathbf{Q}$ see Example 9. However, $\tilde{\mathbf{Q}}$ is a valid core of s since

$$\mathbf{m}_2 \tilde{\mathbf{Q}} \mathbf{b}_2 = \mathbf{m}_2 \begin{bmatrix} \Delta_{1|2} \delta^{-1} \oplus \Delta_{1|2} \oplus \delta^1 \gamma^2 (\delta^1 \gamma^2)^* \Delta_{1|2} \\ \delta^1 \Delta_{1|2} \delta^{-1} \end{bmatrix}.$$

399 Because of Eq. 10 $\Delta_{1|2} \delta^{-1} \oplus \Delta_{1|2} = \Delta_{1|2} (\delta^{-1} \oplus \delta^0) = \Delta_{1|2}$, and therefore

$$\begin{aligned} \mathbf{m}_2 \tilde{\mathbf{Q}} \mathbf{b}_2 &= \left[\Delta_{2|1} \delta^{-1} \Delta_{2|1} \right] \begin{bmatrix} \Delta_{1|2} \oplus \delta^1 \gamma^2 (\delta^1 \gamma^2)^* \Delta_{1|2} \\ \delta^1 \Delta_{1|2} \delta^{-1} \end{bmatrix} \\ &= \Delta_{2|1} \Delta_{1|2} \oplus \Delta_{2|1} \delta^1 \gamma^2 (\delta^1 \gamma^2)^* \Delta_{1|2} \oplus \delta^{-1} \Delta_{2|1} \delta^1 \Delta_{1|2} \delta^{-1} \\ &= \Delta_{2|2} \oplus \delta^1 \Delta_{2|2} \delta^{-1} \oplus \delta^2 \gamma^2 (\delta^2 \gamma^2)^* \Delta_{2|2} = s. \end{aligned}$$

400 To show how the core form can be used to perform basic operations between ultimately
 401 cyclic series in $\mathcal{T}_{per} \llbracket \gamma \rrbracket$ we first elaborate some properties of the \mathbf{m}_ω -vector and \mathbf{b}_ω -vector.
 402 The scalar product $\mathbf{m}_\omega \mathbf{b}_\omega$ of these two vectors is the identity e :

$$\begin{aligned} \mathbf{m}_\omega \otimes \mathbf{b}_\omega &= \delta^0 \Delta_{\omega|1} \Delta_{1|\omega} \delta^{1-\omega} \oplus \cdots \oplus \delta^{1-\omega} \Delta_{\omega|1} \Delta_{1|\omega} \delta^0 \\ &= \delta^0 \Delta_{\omega|\omega} \delta^{1-\omega} \oplus \cdots \oplus \delta^{1-\omega} \Delta_{\omega|\omega} \delta^0 = e, \end{aligned} \tag{24}$$

403 where the latter equality holds because of Proposition 4. The dyadic product $\mathbf{b}_\omega \otimes \mathbf{m}_\omega$ is a
 404 square matrix in $\mathcal{M}_{in}^{ax} \llbracket \gamma, \delta \rrbracket$ denoted by \mathbf{N} . For $i, j \in \{1, \dots, \omega\}$, the entry $(\mathbf{b}_\omega \otimes \mathbf{m}_\omega)_{i,j}$
 405 is given by,

$$(\mathbf{N})_{i,j} = (\mathbf{b}_\omega \otimes \mathbf{m}_\omega)_{i,j} = \Delta_{1|\omega} \delta^{(i-j)+(1-\omega)} \Delta_{\omega|1}.$$

Then, because of $\Delta_{1|\omega}\delta^{-\omega} = \delta^{-1}\Delta_{1|\omega}$ and $\Delta_{1|\omega}\delta^n\Delta_{\omega|1} = \Delta_{1|1} = \mathbf{e}$ for $-\omega < n \leq 0$, see Remark 1, 406
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$$(\mathbf{N})_{i,j} = \begin{cases} \mathbf{e}, & j \leq i, \\ \delta^{-1}, & j > i, \end{cases}$$

i.e., 408

$$\mathbf{N} = \mathbf{b}_\omega \otimes \mathbf{m}_\omega = \begin{bmatrix} \mathbf{e} & \delta^{-1} & \dots & \delta^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \delta^{-1} \\ \mathbf{e} & \dots & \dots & \mathbf{e} \end{bmatrix}. \tag{25}$$

Proposition 8 (Trunk et al. 2018) *The following relations hold:* 409

$$\begin{aligned} \mathbf{N} \oplus \mathbf{I} &= \mathbf{N}, \\ \mathbf{N} \otimes \mathbf{N} &= \mathbf{N}, \\ \mathbf{N} \otimes \mathbf{b}_\omega &= \mathbf{b}_\omega, \\ \mathbf{m}_\omega \otimes \mathbf{N} &= \mathbf{m}_\omega. \end{aligned}$$

4.1 Greatest core matrix 410

From Example 10 it is clear that a series $s \in \mathcal{T}_{per}[\gamma]$ may have several core representations. In the following, we show that a series $s \in \mathcal{T}_{per}[\gamma]$ admits a unique greatest core, denoted $\hat{\mathbf{Q}}$, i.e., $s = \mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega$ and $\hat{\mathbf{Q}} \geq \mathbf{Q}$ for all core matrices \mathbf{Q} such that $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$. Note that, the inequality is in the sense of the dioid $\mathcal{M}_{in}^{ax}[\gamma, \delta]$. This decomposition $s = \mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega$ is particularly useful to compute residuation of series in $\mathcal{T}_{per}[\gamma]$. 411
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Proposition 9 *For $\mathbf{D} \in \mathcal{T}[\gamma]^{1 \times \omega}$ and $\mathbf{P} \in \mathcal{T}[\gamma]^{\omega \times 1}$ one has:* 416

$$\mathbf{m}_\omega \backslash \mathbf{D} = \mathbf{b}_\omega \otimes \mathbf{D}, \tag{26}$$

$$\mathbf{P} \not\backslash \mathbf{b}_\omega = \mathbf{P} \otimes \mathbf{m}_\omega. \tag{27}$$

For $\mathbf{O} \in \mathcal{T}[\gamma]^{n \times \omega}$ and $\mathbf{G} \in \mathcal{T}[\gamma]^{\omega \times n}$ one has: 417

$$(\mathbf{O}\mathbf{N}) \not\backslash \mathbf{m}_\omega = \mathbf{O}\mathbf{N} \otimes \mathbf{b}_\omega, \tag{28}$$

$$\mathbf{b}_\omega \backslash (\mathbf{N}\mathbf{G}) = \mathbf{m}_\omega \otimes (\mathbf{N}\mathbf{G}). \tag{29}$$

Proof By definition, $\mathbf{m}_\omega \backslash \mathbf{D}$ is the greatest solution of inequality 418

$$\mathbf{m}_\omega \otimes \mathbf{X} \leq \mathbf{D}. \tag{30}$$

Clearly since $\mathbf{m}_\omega \mathbf{b}_\omega = \mathbf{e}$, $\mathbf{b}_\omega \mathbf{D}$ satisfies Eq. 30 with equality. It remains to be shown that $\mathbf{b}_\omega \mathbf{D}$ is the greatest solution of Eq. 30. For this, assume that there exists $\mathbf{X}' \geq \mathbf{b}_\omega \mathbf{D}$ solving Eq. 30, i.e., $\mathbf{m}_\omega \mathbf{X}' \leq \mathbf{D}$. Multiplication is order preserving, hence left multiplication by \mathbf{b}_ω results in 419
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$$\mathbf{N} \otimes \mathbf{X}' \leq \mathbf{b}_\omega \mathbf{D}.$$

Furthermore, $\mathbf{X}' \leq \mathbf{N} \otimes \mathbf{X}'$ as $\mathbf{N} = \mathbf{I} \oplus \mathbf{N}$. Hence, $\mathbf{X}' \leq \mathbf{b}_\omega \mathbf{D}$ and therefore $\mathbf{X}' = \mathbf{b}_\omega \mathbf{D}$. This proves that $\mathbf{b}_\omega \mathbf{D}$ is indeed the greatest solution of Eq. 30. Similarly, $\mathbf{X} = \mathbf{P} \mathbf{m}_\omega$ solves $\mathbf{X} \mathbf{b}_\omega \leq \mathbf{P}$ with equality. Suppose $\mathbf{X}' \geq \mathbf{P} \mathbf{m}_\omega$ is a solution, i.e., $\mathbf{X}' \otimes \mathbf{b}_\omega \leq \mathbf{P}$. Right multiplication by \mathbf{m}_ω gives 423
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$$\mathbf{X}' \leq \mathbf{X}' \otimes \mathbf{N} \leq \mathbf{P} \otimes \mathbf{m}_\omega.$$

427 Therefore $X' = P \otimes \mathbf{m}_\omega$ and $P \otimes \mathbf{m}_\omega$ is indeed the greatest solution, and hence
 428 $P \otimes \mathbf{m}_\omega = P \not\leq \mathbf{b}_\omega$. To prove Eq. 28, note that by Proposition 8 $\mathbf{ON} \otimes \mathbf{b}_\omega \otimes \mathbf{m}_\omega =$
 429 \mathbf{ON} . Therefore $\mathbf{ON} \otimes \mathbf{b}_\omega$ is a solution of $X \otimes \mathbf{m}_\omega \leq \mathbf{ON}$. Assume that $X' \geq$
 430 $\mathbf{ON} \otimes \mathbf{b}_\omega$ is another solution, i.e., $X' \mathbf{m}_\omega \leq \mathbf{ON}$. Right multiplication by \mathbf{b}_ω results in
 431 $X' \leq \mathbf{ON} \otimes \mathbf{b}_\omega$. Hence, $\mathbf{ON} \otimes \mathbf{b}_\omega$ is the greatest solution of $X \otimes \mathbf{m}_\omega \leq \mathbf{ON}$ and
 432 $X \otimes \mathbf{m}_\omega \leq \mathbf{ON}$ and $\mathbf{ON} \otimes \mathbf{b}_\omega = \mathbf{ON} \not\leq \mathbf{m}_\omega$. Equation 29 is shown analogously. \square

433 **Proposition 10** Let $\mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega \in \mathcal{T}_{per}[\gamma]$ be a decomposition of $s \in \mathcal{T}_{per}[\gamma]$. The greatest
 434 core matrix is given by $\hat{\mathbf{Q}} = \mathbf{NQN}$.

435 *Proof* Consider the inequality $\mathbf{m}_\omega \tilde{X} \mathbf{b}_\omega \leq s$. Because of Proposition 9, its greatest solution
 436 $\tilde{X} = \mathbf{m}_\omega \not\leq \mathbf{b}_\omega = \mathbf{m}_\omega \not\leq \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega \not\leq \mathbf{b}_\omega$ is given by

$$\tilde{X} = \mathbf{b}_\omega \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega \mathbf{m}_\omega = \mathbf{NQN} = \hat{\mathbf{Q}}.$$

437 Moreover, because of (Proposition 8)

$$\mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega = \mathbf{m}_\omega \mathbf{NQN} \mathbf{b}_\omega = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega = s.$$

438 \square

439 **4.2 Operations between core matrices**

440 To perform addition and multiplication of two ultimately cyclic series $s_1 =$
 441 $\mathbf{m}_{\omega_1} \mathbf{Q}_1 \mathbf{b}_{\omega_1}$, $s_2 = \mathbf{m}_{\omega_2} \mathbf{Q}_2 \mathbf{b}_{\omega_2} \in \mathcal{T}_{per}[\gamma]$ in core form, it is necessary to express the core
 442 matrices $\mathbf{Q}_1 \in \mathcal{M}_{in}^{ax}[\gamma, \delta]^{\omega_1 \times \omega_1}$ and $\mathbf{Q}_2 \in \mathcal{M}_{in}^{ax}[\gamma, \delta]^{\omega_2 \times \omega_2}$ with identical dimensions.
 443 This is possible by expressing both series with their least common period $\omega = lcm(\omega_1, \omega_2)$.

444 **Proposition 11** (Trunk et al. 2018)

445 A series $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega \in \mathcal{T}_{per}[\gamma]$ can be expressed with a multiple period $n\omega$ by extend-
 446 ing the core matrix \mathbf{Q} , i.e., $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega = \mathbf{m}_{n\omega} \mathbf{Q}' \mathbf{b}_{n\omega}$, where $\mathbf{Q}' \in \mathcal{M}_{in}^{ax}[\gamma, \delta]^{n\omega \times n\omega}$ is
 447 given by

$$\mathbf{Q}' = \begin{bmatrix} \Delta_{1|n} \delta^{1-n} \mathbf{NQN} \Delta_{n|1} & \cdots & \Delta_{1|n} \delta^{1-n} \mathbf{NQN} \delta^{1-n} \Delta_{n|1} \\ \vdots & & \vdots \\ \Delta_{1|n} \mathbf{NQN} \Delta_{n|1} & \cdots & \Delta_{1|n} \mathbf{NQN} \delta^{1-n} \Delta_{n|1} \end{bmatrix}.$$

448 **Proposition 12** (Sum of series (Trunk et al. 2018)) Let $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$, $s' = \mathbf{m}_\omega \mathbf{Q}' \mathbf{b}_\omega \in$
 449 $\mathcal{T}_{per}[\gamma]$. Then $s \oplus s' = \mathbf{m}_\omega \mathbf{Q}'' \mathbf{b}_\omega$, where $\mathbf{Q}'' = \mathbf{Q} \oplus \mathbf{Q}'$.

450 **Proposition 13** (Product of series (Trunk et al. 2018)) Let $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega$, $s' = \mathbf{m}_\omega \mathbf{Q}' \mathbf{b}_\omega \in$
 451 $\mathcal{T}_{per}[\gamma]$. Then $s \otimes s' = \mathbf{m}_\omega \mathbf{Q}'' \mathbf{b}_\omega$, where $\mathbf{Q}'' = \mathbf{QNQ}'$.

452 **Proposition 14** (Kleene star of series (Trunk et al. 2018)) Let $s = \mathbf{m}_\omega \mathbf{Q} \mathbf{b}_\omega \in \mathcal{T}_{per}[\gamma]$.
 453 Then,

$$s^* = \mathbf{m}_\omega (\mathbf{QN})^* \mathbf{b}_\omega. \tag{31}$$

454 **Proposition 15** Let $s = \mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega$, $s' = \mathbf{m}_\omega \hat{\mathbf{Q}}' \mathbf{b}_\omega$ be ultimately cyclic series in $\mathcal{T}_{per}[\gamma]$ with
 455 $\hat{\mathbf{Q}}$, respectively $\hat{\mathbf{Q}}'$, their greatest core matrices. Then,

$$s' \not\leq s = \mathbf{m}_\omega (\hat{\mathbf{Q}}' \not\leq \hat{\mathbf{Q}}) \mathbf{b}_\omega, \quad s \not\leq s' = \mathbf{m}_\omega (\hat{\mathbf{Q}} \not\leq \hat{\mathbf{Q}}') \mathbf{b}_\omega$$

Proof

$$\begin{aligned}
 (\mathbf{m}_\omega \hat{\mathbf{Q}}' \mathbf{b}_\omega) \bowtie (\mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega) &= (\hat{\mathbf{Q}}' \mathbf{b}_\omega) \bowtie (\mathbf{m}_\omega \bowtie (\mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega)), \quad (\text{because of (3)}) \\
 &= (\hat{\mathbf{Q}}' \mathbf{b}_\omega) \bowtie (\mathbf{b}_\omega \mathbf{m}_\omega \hat{\mathbf{Q}} \mathbf{b}_\omega), \quad (\text{because of (26)}) \\
 &= (\hat{\mathbf{Q}}' \mathbf{b}_\omega) \bowtie (\hat{\mathbf{Q}} \mathbf{b}_\omega), \quad (\text{as } \mathbf{N} \hat{\mathbf{Q}} = \hat{\mathbf{Q}}, \text{ Proposition 10}) \\
 &= (\hat{\mathbf{Q}}' \mathbf{b}_\omega) \bowtie (\hat{\mathbf{Q}} \not\bowtie \mathbf{m}_\omega) \quad (\text{from (28) and Proposition 10}) \\
 &= \mathbf{b}_\omega \bowtie (\hat{\mathbf{Q}}' \not\bowtie (\hat{\mathbf{Q}} \not\bowtie \mathbf{m}_\omega)), \quad (\text{because of (3)}) \\
 &= \mathbf{b}_\omega \bowtie ((\hat{\mathbf{Q}}' \not\bowtie \hat{\mathbf{Q}}) \not\bowtie \mathbf{m}_\omega) \quad (\text{because of (5)}) \\
 &= \mathbf{m}_\omega (\hat{\mathbf{Q}}' \not\bowtie \hat{\mathbf{Q}}) \mathbf{b}_\omega, \quad (\text{because of Proposition 9}).
 \end{aligned}$$

The proof of the second part of Proposition 15 is analogous. □ 456

Due to Propositions 12, 13, 14 and 15, it is clear that computation of the sum and product, Kleene star operation and product residuation of ultimately cyclic series in $\mathcal{T}_{per}[[\gamma]]$ can be done based on the core of the series, i.e. in the dioid $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. Finally, let us note that this core form of series $s \in \mathcal{T}_{per}[[\gamma]]$ is similar to the core form of series $s \in \mathcal{E}[[\delta]]$, see (Trunk et al. 2017a). More generally the dioid $\mathcal{T}_{per}[[\gamma]]$ with periodic time-operators can be seen as the counter part of the dioid $\mathcal{E}[[\delta]]$, introduced in Cottenceau et al. (2014), with periodic event-operators. The dioid $\mathcal{E}[[\delta]]$ is useful to obtain transfer function matrices for WBTEG. 457
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5 Output reference control 465

In this section, we address the following control problem for TEGs under periodic PS. A reference dater function \bar{z} is given for the output \bar{y} . We want to determine the greatest input dater function \bar{u} that leads to an output $\bar{y} \leq \bar{z}$. The reference dater specifies that the firings of the output transition (which in a manufacturing context, may for example correspond to completion of workpieces) should occur no latter than given instants of time. This has to be achieved by firing the input transition as late as possible. In a manufacturing context, this may correspond to feeding raw material as late as possible. This kind of optimal output reference control is often called "just-in-time" control. For standard TEGs the problem of output reference control was studied in Baccelli et al. (1992), Cohen et al. (1989), Menguy et al. (1998), and Menguy et al. (2000). It is well known for standard TEGs, that the output to an arbitrary input dater function can simply be computed by using the transfer function model $h \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ of the TEG and expressing the input dater as a series $u \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. Then $y = h \otimes u$. Hence, the optimal control problem for standard TEGs simply amounts computing $u_{opt} = h \not\bowtie z$, see Baccelli et al. (1992) and Cohen et al. (1989) for a detailed description. In the following, we show how the earliest response of a TEG under periodic PS can be computed based on its transfer function $h \in \mathcal{T}_{per}[[\gamma]]$ and then how the optimal just-in-time control problem for a TEG under periodic PS can be addressed. For this, we first need to provide some additional algebraic background. 466
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484 **5.1 Subdroids of $\mathcal{T}_{per}[\gamma]$**

485 Recall that an operator $v \in \mathcal{T}$ is called ω -periodic if $\exists \omega \in \mathbb{N}$ such that $\forall k \in \mathbb{Z}_{max}, \mathcal{R}_v(k +$
 486 $\omega) = \omega + \mathcal{R}_v(k)$ (Definition 6) and that the set of ω -periodic T-operators is denoted by \mathcal{T}_ω .
 487 Analogously we say $s \oplus_i v_i \gamma^i \in \mathcal{T}_{per}[\gamma]$ is an ω -periodic series, iff all coefficients are
 488 ω -periodic T-operators, i.e., $\forall i, v_i \in \mathcal{T}_\omega$. The set of ω -periodic series is denoted by $\mathcal{T}_\omega[\gamma]$.

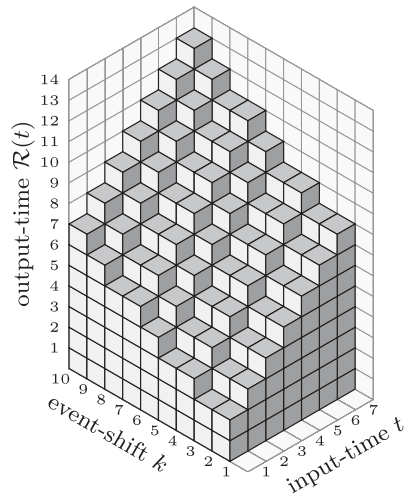
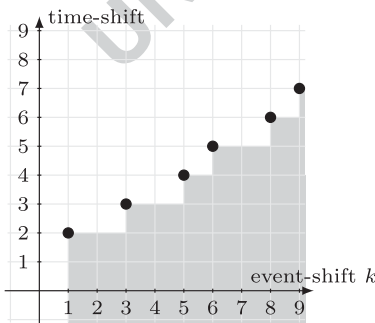
489 **Proposition 16** *The set of ω -periodic series $\mathcal{T}_\omega[\gamma]$ with addition and multiplication given*
 490 *in Definition 8 is a complete subdoid of the dioid $\mathcal{T}_{per}[\gamma]$.*

491 *Proof* According to Propositions 12 and 13, $\mathcal{T}_\omega[\gamma]$ is closed under (infinite) addition and
 492 multiplication. □

493 *Remark 4* The subdoid $\mathcal{T}_1[\gamma]$ of $\mathcal{T}_{per}[\gamma]$, i.e. the set of 1-periodic series, is the dioid
 494 $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. Moreover, as any 1-periodic series is also ω -periodic ($\omega \in \mathbb{N}$), $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$
 495 is subdoid of $\mathcal{T}_\omega[\gamma]$ for any $\omega \in \mathbb{N}$.

496 Due to the subdoid structure of $\mathcal{T}_{per}[\gamma]$, one can define the canonical injection $\text{Inj} :$
 497 $\mathcal{M}_{in}^{ax}[[\gamma, \delta]] \rightarrow \mathcal{T}_{per}[\gamma]$, with $\text{Inj}(x) = x$. For a graphical illustration of this canonical
 498 injection see the following example.

499 *Example 11* Let us consider the series $s = \gamma^1 \delta^2 \oplus (\gamma^3 \delta^3 \oplus \gamma^5 \delta^4) (\gamma^3 \delta^2)^* \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$,
 500 with a graphical representation given in Fig. 8a. The graphical representation of the canon-
 501 ical injection $\text{Inj}(s) \in \mathcal{T}_{per}[\gamma]$ is shown in Fig. 8b. The series $s \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ (Fig. 8a)



(a) Graphical representation of $s \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. (b) Graphical representation of $\text{Inj}(s) \in \mathcal{T}_{per}[\gamma]$.

Fig. 8 Illustration of the canonical injection $\text{Inj} : \mathcal{M}_{in}^{ax}[[\gamma, \delta]] \rightarrow \mathcal{T}_{per}[\gamma]$ of the series $s = \gamma^1 \delta^2 \oplus (\gamma^3 \delta^3 \oplus \gamma^5 \delta^4) (\gamma^3 \delta^2)^* \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$

corresponds to the event-shift/output-time plane for the input-time value 0 of the 3D representation of the series $\text{Inj}(s) \in \mathcal{T}_{per}[\gamma]$ (Fig. 8b). Moreover, the canonical injection $\text{Inj}(s) \in \mathcal{T}_{per}[\gamma]$ is 1-periodic, this means the coefficients v_i of γ^i are 1-periodic, i.e., $\mathcal{R}_{v_i}(t)$ are quasi 1-periodic. Therefore, the event-shift/output-time plane for the input-time value 1 corresponds to the series $\delta^1 s \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ and for the input-time value 2 to the series $\delta^2 s \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$, etc.

Lemma 1 *Let $v\gamma^n \in \mathcal{T}_\omega[\gamma]$ be an ω -periodic monomial. Then the residual $\text{Inj}^\sharp(v\gamma^n)$ is given by*

$$\text{Inj}^\sharp(v\gamma^n) = \delta^{t=0, \dots, \omega-1} \min_{(\mathcal{R}_v(t)-t)} \gamma^n. \tag{32}$$

Proof By definition, $\text{Inj}^\sharp(v\gamma^n)$ is the greatest solution $x \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ of the following inequality:

$$v\gamma^n \geq \text{Inj}(x) = \text{Inj}\left(\bigoplus_i \gamma^{n_i} \delta^{\zeta_i}\right) = \bigoplus_i \gamma^{n_i} \delta^{\zeta_i}. \tag{33}$$

Clearly, the least n_i such that inequality Eq. 33 holds are n and thus,

$$v\gamma^n \geq \bigoplus_i (\gamma^n \delta^{\zeta_i}) = \gamma^n \delta^\tau. \tag{34}$$

where the latter equality holds for $\tau = \max_i(\zeta_i)$, because of Eq. 20. Since the inequality $v\gamma^n \geq \gamma^n \delta^\tau$ in $\mathcal{T}_\omega[\gamma]$ holds iff the inequality $v \geq \delta^\tau$ in \mathcal{T}_ω holds, it remains to find the greatest τ such that $v \geq \delta^\tau$ holds. By considering the isomorphism between T-operators and release-time functions, see Eq. 13, this is equivalent to $\mathcal{R}_v(t) \geq \mathcal{R}_{\delta^\tau}(t)$, $\forall t \in \mathbb{Z}_{max}$.

By using $\mathcal{R}_{\delta^\tau}(t) = \tau + t$, see Eq. 7, one obtains

$$\mathcal{R}_v(t) \geq \tau + t \Leftrightarrow \tau \leq \mathcal{R}_v(t) - t, \quad \forall t \in \mathbb{Z}_{max}. \tag{35}$$

Since \mathcal{R}_v is a quasi ω -periodic function it is sufficient to evaluate the function for $\forall t \in \{0, \dots, \omega - 1\}$. Therefore the greatest τ such that Eq. 35 (resp. Eq. 34) holds is

$$\tau = \min_{t=0, \dots, \omega-1} (\mathcal{R}_v(t) - t).$$

□ 520

Lemma 1 can be extended to arbitrary series in $\mathcal{T}_\omega[\gamma]$. To do this, note that the canonical representation in Proposition 6 can be generalized to infinite sums.

Proposition 17 *Let $s = \bigoplus_i v_i \gamma^{n_i} \in \mathcal{T}_\omega[\gamma]$ be an ω -periodic series in canonical representation. Then*

$$\text{Inj}^\sharp(s) = \bigoplus_i \delta^{t=0, \dots, \omega-1} \min_{(\mathcal{R}_{v_i}(t)-t)} \gamma^{n_i}. \tag{36}$$

Proof Consider $s = \bigoplus_i v_i \gamma^{n_i}$ in the canonical form, i.e., $n_i < n_{i+1}$ and $v_i < v_{i+1}$ and let \mathcal{R}_{v_i} be the release-time function associated with v_i . Recall that $\text{Inj}^\sharp(s)$ is the greatest solution x in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$ of inequality $\text{Inj}(x) \leq s$. This is given by $\bigoplus_i \delta^{\tau_i} \gamma^{n_i}$ where τ_i is the greatest integer such that $\delta^{\tau_i} \leq v_i$. Repeating the first step of the proof of Lemma 1, this is given by $\tau_i = \min_{t=0, \dots, \omega-1} (\mathcal{R}_{v_i}(t) - t)$. □ 529

530 **5.1.1 Zero slice mapping $\Psi_\omega : \mathcal{T}_\omega[[\gamma]] \rightarrow \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$**

531 Recall that $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ is a subdioid of $\mathcal{T}_\omega[[\gamma]]$, hence we can define a specific projection
 532 from $\mathcal{T}_\omega[[\gamma]]$ into $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ as follows.

533 **Definition 10** Let $s = \bigoplus_i v_i \gamma^{n_i} \in \mathcal{T}_\omega[[\gamma]]$ be an ω -periodic series, then

$$\Psi_\omega(s) = \Psi_\omega\left(\bigoplus_i v_i s \gamma^{n_i}\right) = \bigoplus_i \gamma^{n_i} \delta^{\mathcal{R}_{v_i}(0)}. \tag{37}$$

534 This projection Ψ_ω has an intuitive graphical interpretation. For a given $s \in \mathcal{T}_\omega[[\gamma]]$ the
 535 series $\Psi_\omega(s) \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ corresponds to the slice in the event/output-time plane of the
 536 3D representation of $s \in \mathcal{T}_\omega[[\gamma]]$ at the input-time value 0. Thus, this projection is also
 537 called zero-slice mapping.

538 *Example 12* Consider the polynomial $p = (\delta^1 \Delta_{4|4} \delta^{-1} \oplus \delta^{-2} \Delta_{4|4}) \gamma^0 \oplus (\delta^5 \Delta_{4|4} \delta^{-1} \oplus$
 539 $\delta^2 \Delta_{4|4}) \gamma^2 \oplus (\delta^5 \Delta_{4|4} \oplus \delta^6 \Delta_{4|4} \delta^{-1}) \gamma^4 \in \mathcal{T}_{per}[[\gamma]]$ from Example 7 with graphical
 540 representation given in Fig. 6. Then,

$$\Psi_4(p) = \delta^1 \gamma^0 \oplus \delta^5 \gamma^2 \oplus \delta^6 \gamma^4.$$

541 The series $\Psi_4(p)$ corresponds to the slice in the (event-shift/output-time)-plane for the
 542 input-time value $t = 0$ in the 3D representation of p , see Fig. 9a and b.

543 The projection Ψ_ω is by definition lower-semicontinuous, therefore Ψ_ω is residuated.

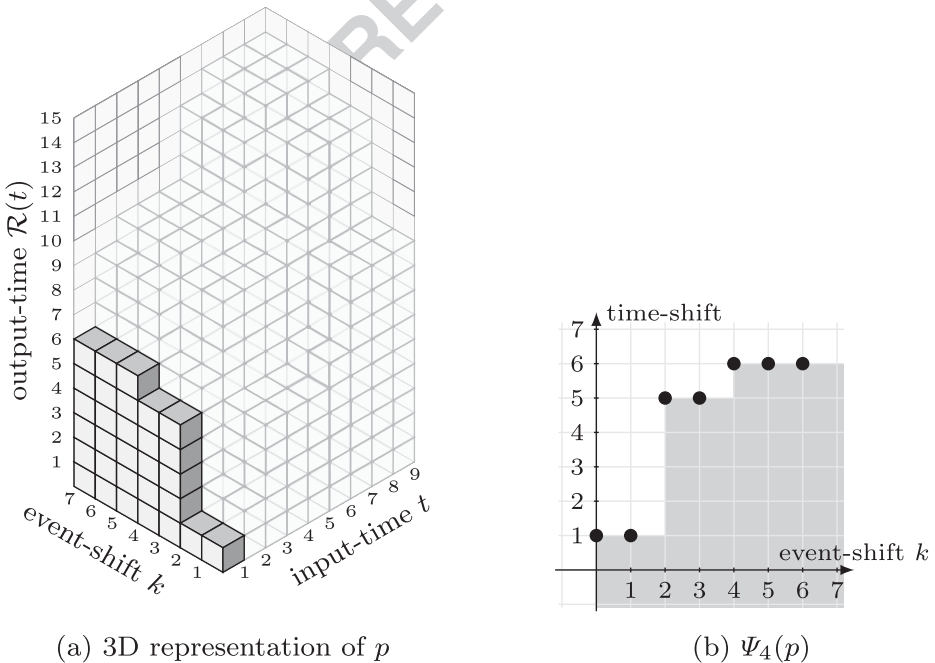


Fig. 9 Illustration of Projection $\Psi_4(p)$

Proposition 18 Let $s = \bigoplus_i \gamma^{n_i} \delta^{\tau_i} \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. The residual $\Psi_\omega^\sharp(s) \in \mathcal{T}_\omega [[\gamma]]$ of s is 544

$$\Psi_\omega^\sharp(s) = \bigoplus_i \gamma^{n_i} \delta^{\tau_i} \Delta_{\omega|\omega} = s \Delta_{\omega|\omega}. \tag{38}$$

Proof By definition of the residuated mapping, $\Psi_\omega^\sharp(s) \in \mathcal{T}_\omega [[\gamma]]$ is the greatest solution of inequality 545
546

$$s = \bigoplus_i \gamma^{n_i} \delta^{\tau_i} \geq \Psi_\omega(x). \tag{39}$$

We first show that Eq. 38 satisfies Eq. 39 with equality. 547

$$\Psi_\omega \left(\bigoplus_i \gamma^{n_i} \delta^{\tau_i} \Delta_{\omega|\omega} \right) = \bigoplus_i \gamma^{n_i} \delta^{\mathcal{R}_{\delta^{\tau_i} \Delta_{\omega|\omega}}(0)} = \bigoplus_i \gamma^{n_i} \delta^{\tau_i},$$

since $\mathcal{R}_{\delta^{\tau_i} \Delta_{\omega|\omega}}(0) = \tau_i + \lceil 0/\omega \rceil \omega = \tau_i$, see Eqs. 7 and 9. Taking into account that Ψ_ω is isotone, it remains to show that $\bigoplus_i \gamma^{n_i} \delta^{\tau_i} \Delta_{\omega|\omega}$ is the greatest solution of 548
549

$$s = \bigoplus_i \gamma^{n_i} \delta^{\tau_i} = \Psi_\omega(x) \tag{40}$$

For this, let $x = \bigoplus_j v_j \gamma^{n_j}$ be an arbitrary series in $\mathcal{T}_\omega [[\gamma]]$. Then $\Psi_\omega(x) = \bigoplus_j \gamma^{n_j} \delta^{\mathcal{R}_{v_j}(0)}$. 550
Clearly, to achieve equality we need $n_j = n_i$ and $\mathcal{R}_{v_j}(0) = \tau_i$. Furthermore, we are looking 551
for the greatest $v_j \in \mathcal{T}_\omega$, such that $\tau_i = \mathcal{R}_{v_j}(0)$. Due to the canonical form (Proposition 552
3) we can write an ω -periodic T-operator v_j as $\bigoplus_{i=1}^\omega \delta^{\zeta_i} \Delta_{\omega|\omega} \gamma^{\zeta'_i}$ with $-\omega < \zeta'_i \leq 0$. This 553
operator corresponds to the release-time function 554

$$\mathcal{R}_{v_j}(t) = \max_{i=1, \dots, \omega} \left(\zeta_i + \left\lceil \frac{\zeta'_i + t}{\omega} \right\rceil \omega \right).$$

Now we examine $\mathcal{R}_{v_j}(t)$ for $t = 0$, thus 555

$$\mathcal{R}_{v_j}(0) = \max_{i=1, \dots, \omega} \left(\zeta_i + \left\lceil \frac{\zeta'_i}{\omega} \right\rceil \omega \right).$$

Recall that $-\omega < \zeta'_i \leq 0$, hence $\mathcal{R}_{v_j}(t) = \tau_i + \lceil (0 + t)/\omega \rceil \omega$ is the greatest quasi ω - 556
periodic release-time function such that $\mathcal{R}_{v_j}(0) = \tau_i$. The corresponding greatest T-operator 557
is accordingly $\delta^{\tau_i} \Delta_{\omega|\omega}$. \square 558

5.1.2 Dater functions and series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ 559

A convenient way to compute the output of a TEG under periodic PS is to express its input 560
and output dater functions as series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. The following proposition gives a link 561
between dater functions and series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. 562

Proposition 19 (Baccelli et al. 1992) A dater function $\bar{d} : \mathbb{Z} \rightarrow \mathbb{Z}_{max}$ can be expressed as 563
a series $d \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, such that, 564

$$d = \left(\bigoplus_{\{k \in \mathbb{Z} | -\infty < \bar{d} < +\infty\}} \gamma^k \delta^{\bar{d}} \right) \oplus \left(\bigoplus_{\{k \in \mathbb{Z} | \bar{d} = +\infty\}} \gamma^k \delta^* \right). \tag{41}$$

For a more detailed description of the link between dater functions and the associated 565
series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, see e.g. Baccelli et al. (1992) and Cohen et al. (1991). The impulse is 566
a specific dater function, namely $\mathcal{I}(k) = -\infty$ if $k < 0$ and 0 otherwise. Hence, an impulse 567

568 as the input of a TEG corresponds to an infinity of firings of its input transition at time 0.
 569 The $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ series corresponding to an impulse is the unit element $e \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$,
 570 see Baccelli et al. (1992) and Cohen et al. (1991).

571 Moreover, a dater function \bar{d} and its series representation $d \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ are related by

$$\bar{d}(k) = (d\mathcal{I})(k)$$

572 The impulse response of a TEG can be readily expressed via the TEG transfer function
 573 $h \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. The dater function $\bar{y}_{\mathcal{I}}$ the impulse response is characterized by

$$\bar{y}_{\mathcal{I}}(k) = (h\mathcal{I})(k)$$

574 while the corresponding series is obtained by

$$y_{\mathcal{I}} = h \otimes e = h.$$

575 Similarly, the response to an arbitrary input series u (with dater function \bar{u}) is

$$y = h \otimes u,$$

576 respectively

$$\bar{y}(k) = (h\bar{u})(k) = (h(u\mathcal{I}))(k).$$

577 In contrast, the transfer function $h \in \mathcal{T}_{per}[[\gamma]]$ of a TEG under periodic PS is not entirely
 578 characterized by the impulse response. As the impulse corresponds to an infinity of firings
 579 at time 0, the impulse response of a TEG under periodic PS is characterized by the slice
 580 in the (event-shift/output-time)-plane at the input-time value 0 of the 3D representation of
 581 its transfer function $h \in \mathcal{T}_{per}[[\gamma]]$, see e.g., Example 12. Hence, for a TEG under periodic
 582 PS with transfer function h , the impulse response $\bar{y} = (h\mathcal{I})$ corresponds to the series $y =$
 583 $\Psi_{\omega}(h) \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, see Definition 10 and Example 12. It should be clear that in contrast
 584 to standard TEGs, the impulse response of a TEG under periodic PS only provides partial
 585 information of its transfer function. For TEGs under periodic PS, the above duality between
 586 representing the output as counter function and series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ reads as follows. Let
 587 $h \in \mathcal{T}_{per}[[\gamma]]$ be the transfer function of the TEG under periodic PS and $\bar{u} \in \Sigma$, respectively
 588 $u \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, be the input. Then we obtain the output counter function $\bar{y} \in \Sigma$ by

$$\bar{y}(k) = (h\bar{u})(k),$$

589 and the corresponding output series $y \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ by

$$y = \Psi_{\omega}(h \otimes \text{Inj}(u)). \tag{42}$$

590 *Example 13* Recall the simple supply chain in Example 2 with the TEGPS model shown in
 591 Fig. 2. The transfer function is $h = \delta^{11}(\gamma^2\delta^{20})^* \Delta_{20|20}\delta^{-1}$. This transfer function was com-
 592 puted with the ETVO toolbox (Cottenceau et al. 2019) available online at: <http://perso-laris.univ-angers.fr/~cottenceau/etvo.html>, this toolbox implements the algorithms given in this
 593 section. Moreover, consider the following input dater function:

$$\bar{u}(k) = \begin{cases} -\infty & \text{for } k < 0; \\ 0 & \text{for } k = 0; \\ 5 & \text{for } k = 1, 2; \\ 35 & \text{for } k = 3, 4, 5, 6; \\ \infty & \text{for } k \geq 7. \end{cases} \tag{43}$$

595 This dater function is interpreted as follows: the first product available for transport from
 596 factory 1 to factory 2 is ready at time instant 0. The second and third at time instant 5. The
 597 4th, 5th, 6th and 7th at time instant 35. According to Eq. 41, the series $u \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$

corresponding to this dater function is $u = \gamma^0\delta^0 \oplus \gamma^1\delta^5 \oplus \gamma^3\delta^{35} \oplus \gamma^7\delta^*$. The output $y \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ of the system is computed as

$$\begin{aligned}
 y &= \Psi_\omega (h \otimes \text{Inj}(u)) \\
 &= \Psi_\omega \left(\delta^{11}(\gamma^2\delta^{20})^* \Delta_{20|20}\delta^{-1} \otimes (\gamma^0\delta^0 \oplus \gamma^1\delta^5 \oplus \gamma^3\delta^{35} \oplus \gamma^7\delta^*) \right) \\
 &= \Psi_\omega \left(\delta^{11} \Delta_{20|20}\delta^{-1} (\gamma^2\delta^{20})^* \otimes (\gamma^0\delta^0 \oplus \gamma^1\delta^5 \oplus \gamma^3\delta^{35} \oplus \gamma^7\delta^*) \right) \\
 &= \Psi_\omega \left((\delta^{11} \Delta_{20|20}\delta^{-1} \oplus \delta^{31} \Delta_{20|20}\delta^{-16} \gamma^1 \oplus \delta^{51} \Delta_{20|20}\delta^{-6} \gamma^3) (\gamma^2\delta^{20})^* \right. \\
 &\quad \left. \oplus \delta^{11} \Delta_{20|20}\delta^{-1} \gamma^7 (\gamma^2\delta^2)^* \delta^* \right) \\
 &= \Psi_\omega \left((\delta^{11} \Delta_{20|20}\delta^{-1} \oplus \delta^{31} \Delta_{20|20}\delta^{-16} \gamma^1 \oplus \delta^{51} \Delta_{20|20}\delta^{-6} \gamma^3) (\gamma^2\delta^{20})^* \right. \\
 &\quad \left. \oplus \delta^{11} \Delta_{20|20}\delta^{-1} \gamma^7 \delta^* \right) \\
 &= (\delta^{11} \oplus \delta^{31} \gamma^1 \oplus \delta^{51} \gamma^3) (\gamma^2\delta^{20})^* \oplus \delta^{11} \delta^* \gamma^7 \\
 &= (\delta^{11} \oplus \delta^{31} \gamma^1 \oplus \delta^{51} \gamma^3 \oplus \delta^{71} \gamma^5 \oplus \delta^{91} \gamma^7 \oplus \dots) \oplus \delta^{11} \delta^* \gamma^7 \\
 &= \delta^{11} \oplus \delta^{31} \gamma^1 \oplus \delta^{51} \gamma^3 \oplus \delta^{71} \gamma^5 \oplus \delta^* \gamma^7,
 \end{aligned}$$

with associated dater function \bar{y}

$$\bar{y}(k) = \begin{cases} -\infty & \text{for } k < 0; \\ 11 & \text{for } k = 0; \\ 31 & \text{for } k = 1, 2; \\ 51 & \text{for } k = 3, 4; \\ 71 & \text{for } k = 5, 6; \\ \infty & \text{for } k \geq 7. \end{cases}$$

Hence, this implies that the first product is available at factory 2 at time instant 11, the second and third at time instant 31, the 4th and 5th at time instant 51, and the 6th and 7th at time instant 71.

5.2 Optimal Output Reference Control

The optimal output reference control problem for a TEG under periodic PS with a transfer function $h \in \mathcal{T}_{per} [[\gamma]]$ is to find the greatest input dater \bar{u} such that, $\forall k \in \mathbb{Z}$

$$\bar{z}(k) \geq (h\bar{u})(k), \tag{44}$$

where \bar{z} is a given reference dater.

If, instead, we represent the unknown input and the reference as series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, Eq. 44 is rephrased as

$$z \geq \Psi_\omega (h \otimes \text{Inj}(u)), \tag{45}$$

where the series $z, u \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ correspond to the dater functions \bar{z} and \bar{u} .

Theorem 3 Let $h \in \mathcal{T}_{per} [[\gamma]]$ be the transfer function of a single-input single-output (SISO) TEG under periodic PS and $z \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ a given output reference for the system, then the optimal input u_{opt} , i.e., the greatest solution of Eq. 45, is

$$u_{opt} = \text{Inj}^\# (h \setminus \Psi_\omega^\# (z)). \tag{46}$$

614 *Proof* As Ψ_ω is a residuated mapping (see Proposition 18), Eq. 46 is equivalent to $h \otimes$
 615 $\text{Inj}(u) \preceq \Psi_\omega^\sharp(z)$. This, in turn, is equivalent to $\text{Inj}(u) \preceq h \backslash \Psi_\omega^\sharp(z)$ as left multiplication in
 616 $\mathcal{T}_{per}[\gamma]$ is residuated. Finally as Inj is residuated (Proposition 17), the greatest solution of
 617 the latter inequality is Eq. 46. \square

618 Equation 46 is often referred to as the just-in-time solution. Note that the notation of
 619 greatest is in the sense of the order \succeq in the dioid $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$.

620 *Example 14* Recall the supply chain of Example 2, which is modelled by the TEG under
 621 periodic PS given in Fig. 2 and has transfer function

$$h = \delta^{11} \Delta_{20|20} \delta^{-1} (\gamma^2 \delta^{20})^* \in \mathcal{T}_{per}[\gamma].$$

622 Let us consider the following dater function (see Fig. 10), which describes at which instants
 623 of time goods from factory 1 need to be available at factory 2 at the latest.

$$\bar{z}(k) = \begin{cases} -\infty & \text{for } k < 0, \\ 25 & \text{for } k = 0, 1, \\ 45 + 15j & \text{for } k = 2 + j \text{ with } j \in \mathbb{N}_0. \end{cases}$$

624 The control problem is now, to compute \bar{u} , i.e. the maximal time when goods from
 625 factory 1 are ready to be shipped to factory 2, such that Eq. 44 respectively Eq. 45,
 626 holds. To apply Eq. 46, the dater function \bar{z} is expressed by the series $z = \delta^{25} \oplus$
 627 $\gamma^2 \delta^{45} (\gamma^1 \delta^{15})^* \in \mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. Then according to 18, $\Psi_{20}^\sharp(z) = z \Delta_{20|20} = \delta^{25} \Delta_{20|20} \oplus$
 628 $(\gamma^1 \delta^{15})^* (\gamma^2 \delta^{45} \Delta_{20|20})$ and

$$u_{opt} = \text{Inj}^\sharp(h \backslash \Psi_{20}^\sharp(z)) = (\delta^1 \oplus \gamma^2 \delta^{21} \oplus \gamma^3 \delta^{41}) (\gamma^4 \delta^{60})^*.$$

629 where the latter equality has been computed using ETVO toolbox (Cottenceau et al. 2019).
 630 The response y of the TEGPS to the optimal input u_{opt} is

$$y = \Psi_2(h \otimes \text{Inj}(u_{opt})) = (\delta^{11} \oplus \gamma^2 \delta^{31} \oplus \gamma^3 \delta^{51}) (\gamma^4 \delta^{60})^*.$$

631 This series corresponds to the dater function,

$$\bar{y}(k) = \begin{cases} -\infty & \text{for } k < 0, \\ 11 + 60j & \text{for } k = 4j \text{ and } k = 1 + 4j \text{ with } j \in \mathbb{N}_0, \\ 31 + 60j & \text{for } k = 2 + 4j \text{ with } j \in \mathbb{N}_0, \\ 51 + 60j & \text{for } k = 3 + 4j \text{ with } j \in \mathbb{N}_0. \end{cases}$$

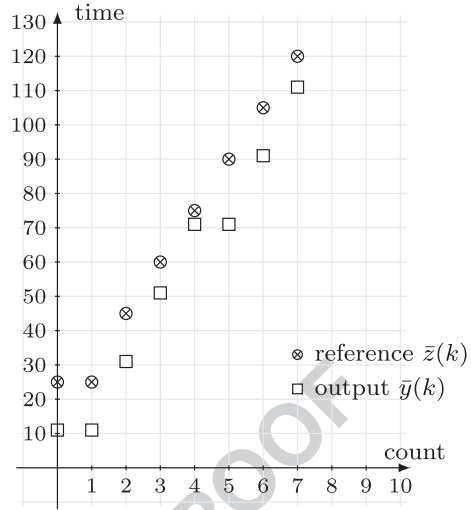
632 Figure 10 illustrates the output reference \bar{z} and \bar{y} resulting from the optimal input \bar{u}_{opt} .
 633 Clearly, as required, $\bar{z} \succeq \bar{y}$. This means, the goods are shipped from factory 1 as late as
 634 possible, but arrive in factory 2 in time to meet the production deadlines there.

635 *Remark 5* Output reference control can be readily extended to multiple-input multiple-
 636 output (MIMO) TEGs under periodic PS. In this case the earliest behaviour of a TEG under
 637 periodic PS is modeled by a transfer function matrix $\mathbf{H} \in \mathcal{T}_{per}[\gamma]^{p \times g}$. Then the optimal
 638 output reference control problem is, for all $j = 1, \dots, p$,

$$z_j \succeq \Psi_\omega \left(\bigoplus_{i=1}^g (\mathbf{H})_{j,i} \text{Inj}(u_i) \right), \tag{47}$$

639

Fig. 10 Output reference \bar{z} and system response \bar{y} to optimal input \bar{u}_{opt}



where $z_j \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ represents the reference for the j^{th} output and $u_i \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$ is i^{th} input of the system. As Ψ_ω is a lower semi-continuous mapping we can write Eq. 47 as, for $j = 1, \dots, p$,

$$z_j \succeq \bigoplus_{i=1}^g \Psi_\omega((\mathbf{H})_{j,i} \text{Inj}(u_i)), \tag{48}$$

The latter set of p inequalities, can be written as a set of $p * q$ simpler inequalities, i.e., $\forall j \in \{1, \dots, p\}$ and $\forall i \in \{1, \dots, g\}$,

$$z_j \succeq \Psi_\omega((\mathbf{H})_{j,i} \text{Inj}(u_i)). \tag{49}$$

Observe that each of these inequalities has the form of Eq. 45. Hence, the optimal i^{th} input $u_{i,opt} \in \mathcal{M}_{in}^{ax} [[\gamma, \delta]]$, i.e., the greatest u_i that satisfies Eq. 49 for $j = 1, \dots, p$, is

$$u_{i,opt} = \bigwedge_{j=1}^p \left(\text{Inj}^\#((\mathbf{H})_{j,i} \Psi_\omega^\#(z_j)) \right).$$

Hence, the only difference to the SISO case is an additional infimum operation between series in $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$.

6 Conclusion

In this paper, we have introduced algebraic tools to obtain transfer function matrices for a subclass of Timed Event Graphs under Partial Synchronization, namely the case where partial synchronization of transitions is characterized by periodic signals. We have introduced the dioid $\mathcal{T}_{per} [[\gamma]]$, which is a quotient dioid of formal power series in γ with coefficients that are periodic time-operators. We have shown that all relevant operations on ultimately cyclic series s in this dioid can be reduced to operations on matrices in the subdioid $\mathcal{M}_{in}^{ax} [[\gamma, \delta]]$. An advantage of this approach is that existing software tools for

657 standard TEGs in the dioid $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$, e.g. Hardouin et al. (2009) can be applied to the
658 more general class of TEGsPS with periodic PS. The more recent toolbox (Cottenceau et al.
659 2019), based also on Hardouin et al. (2009), implements the "translation process" from
660 $\mathcal{T}_{per}[[\gamma]]$ to $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$. Moreover, based on transfer functions for this class of TEGsPS
661 we have solved the corresponding optimal output reference control problem. In particular,
662 the proposed control method provides the optimal control input under the "just-in-time" cri-
663 terion. One possible extension of this work is to modify the control strategy such that online
664 updates of the reference trajectory can be considered. This would allow the system to react
665 to a change in customer demands, and will be considered in future work.

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