Few-cycle solitons in supercontinuum generation dynamics

Hervé Leblond, Philippe Grelu, Dumitru Mihalache

Non-SVEA models for supercontinuum generation

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Introduction

- The shortest laser pulses: a duration of a few optical cycles.

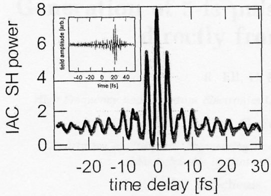


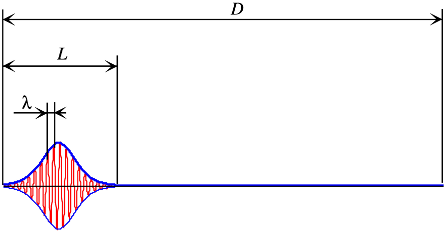
Fig. 4. Measured and reconstructed IAC of the pulse. A phase-retrieval algorithm reveals a pulse width of 5 fs. The reconstructed IAC fits perfectly and is not distinguishable. The reconstructed electric field is displayed in the inset. SH, second harmonic.

Autocorrelation trace, R. Ell et al., Optics Letters 26 (6), 373 (2001).

- Pulse duration down to a few fs. Ex. above: $5\,\mathrm{fs} = 5 \times 10^{-15}\mathrm{s}$.
- How to model the propagation of such pulses?

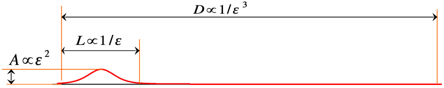
Solitary wave vs envelope solitons

- Envelope soliton: the usual optical soliton in the ps range



- Pulse duration $L \gg \lambda$ wavelength
- Typical model: NonLinear Schrödinger equation (NLS)

- Solitary wave soliton: the hydrodynamical soliton



- A single oscillation
- Typical model: Korteweg-de Vries equation (KdV)

- Few-cycle optical solitons: $L \sim \lambda$
  - The slowly varying envelope approximation is not valid
  - Generalized NLS equation
  - We seek a different approach based on KdV-type models

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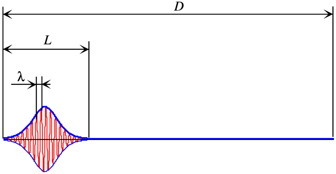

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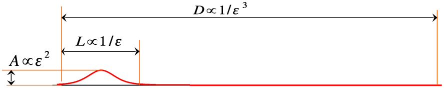

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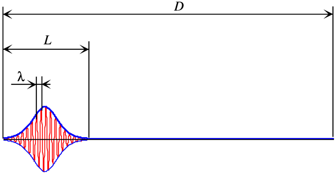

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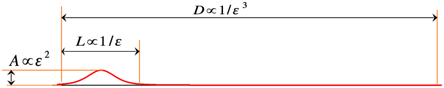

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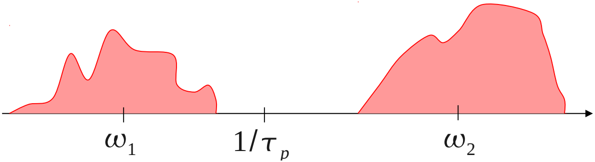
A transparent medium

- The general spectrum of a transparent medium



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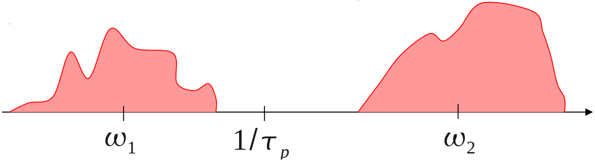

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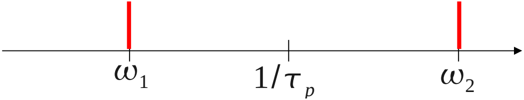

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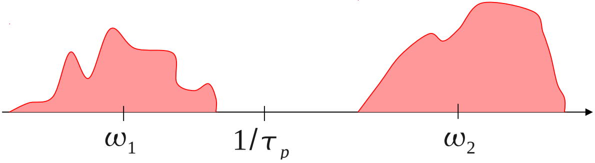

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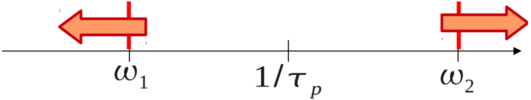

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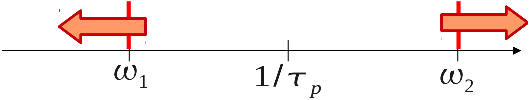

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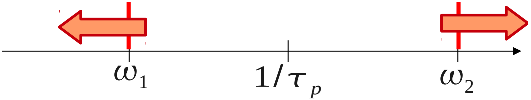

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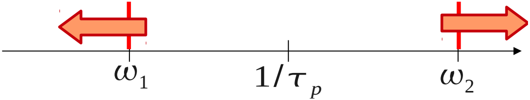

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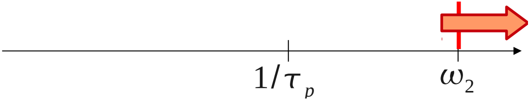

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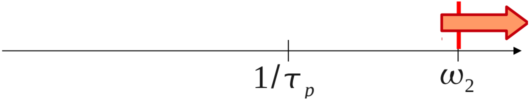

$\Longrightarrow$ Long-wave approximation
modified Korteweg-de Vries (mKdV) equation

$$\frac{\partial E}{\partial \zeta} = \frac{1}{6} \frac{d^3 k}{d\omega^3}\bigg|_{\omega=0} \frac{\partial^3 E}{\partial \tau^3} - \frac{6\pi}{nc} \chi^{(3)}(\omega; \omega, \omega, -\omega)\bigg|_{\omega=0} \frac{\partial E^3}{\partial \tau}$$

H. Leblond and F. Sanchez, *Phys. Rev. A* **67**, 013804 (2003)

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  $\Longrightarrow$ Short-wave approximation
- sine-Gordon (sG) equation: $\dfrac{\partial^2 \psi}{\partial z \partial t} = c_1 \sin \psi$

  with $c_1 = \dfrac{w_\infty}{w_r}$: normalized initial population difference

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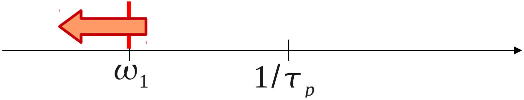

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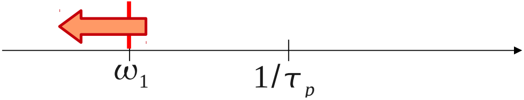

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- Then the two approximations are brought together to yield a general model:

The mKdV-sG equation

$$\frac{\partial^2 \psi}{\partial z \partial t} + c_1 \sin \psi + c_2 \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial t} \right)^3 + c_3 \frac{\partial^4 \psi}{\partial t^4} = 0$$

- $\frac{\partial u}{\partial z} + c_1 \sin \int^t u + c_2 \frac{\partial u^3}{\partial t} + c_3 \frac{\partial^3 u}{\partial t^3} = 0$

with $u = \frac{\partial \psi}{\partial t} = \frac{E}{E_r}$: normalized electric field

- Integrable by inverse scattering transform in some cases:

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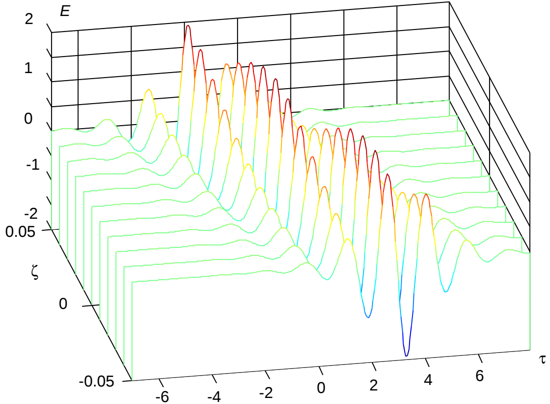
- $$\frac{\partial u}{\partial z} + c_1 \sin \int^t u + c_2 \frac{\partial u^3}{\partial t} + 2c_2 \frac{\partial^3 u}{\partial t^3} = 0$$

  with $u = \frac{\partial \psi}{\partial t} = \frac{E}{E_r}$: normalized electric field

- Integrable by inverse scattering transform in some cases:

  mKdV, sG, and $c_3 = 2c_2$.

The analytical breather solution



- A few cycle soliton:
  - Not spread out by dispersion
  - Stable
  - However, oscillates (breather)

H. Leblond, S.V. Sazonov, I.V. Mel'nikov, D. Mihalache, and F. Sanchez, *Phys. Rev. A* **74**, 063815 (2006)

More than two atomic levels

- UV transitions:
  mKdV model generalizes without modification to a general Hamiltonian.

  H. Triki, H. Leblond, D. Mihalache, *Opt. Comm.* **285**, 3179-3186 (2012)
- IR transitions: Not so simple
- The sG model
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- Generalization: one w for each transition.

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\begin{aligned}
\frac{\partial E}{\partial z} &= \frac{-N}{\varepsilon_0 c}\Omega Q \\
\hbar\frac{\partial w}{\partial t} &= -EQ \\
\hbar\frac{\partial Q}{\partial t} &= |\mu|^2 Ew
\end{aligned}
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identical to Self Induced Transparency equations,
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More than two atomic levels

- IR transitions: Not so simple
- The sG model for 2 transitions (4 level), generalizes to:

$$\begin{aligned}
\frac{\partial E}{\partial z} &= \frac{-N\Omega}{\varepsilon_0 c}(\Omega_1 Q_1 + \Omega_2 Q_2) \\
\hbar\frac{\partial w_j}{\partial t} &= -EQ_j, \qquad j = 1,2 \\
\hbar\frac{\partial Q_j}{\partial t} &= |\mu_j|^2 E w_j, \qquad j = 1,2
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- Generalization: one w for each transition.

- The above system reduces to

$$\frac{\partial u}{\partial z} + c_1 \sin \int^{\tau} u d\tau' + q c_1 \sin \nu \int^{\tau} u d\tau' = 0$$

$\longrightarrow$ a generalized double sine-Gordon equation

- Both admit breather solutions:

H. Triki, H. Leblond, D. Mihalache, *Phys. Rev. A* **86**, 063825 (2012)

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$$\frac{\partial u}{\partial z} + c_1 \sin \int^{\tau} u d\tau' + q c_1 \sin 2 \int^{\tau} u d\tau' = 0$$

$\longrightarrow$ a generalized double sine-Gordon equation

- Under certain conditions: the double sine Gordon equation ($\nu = 2$).
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H. Triki, H. Leblond, D. Mihalache, *Phys. Rev. A* **86**, 063825 (2012)

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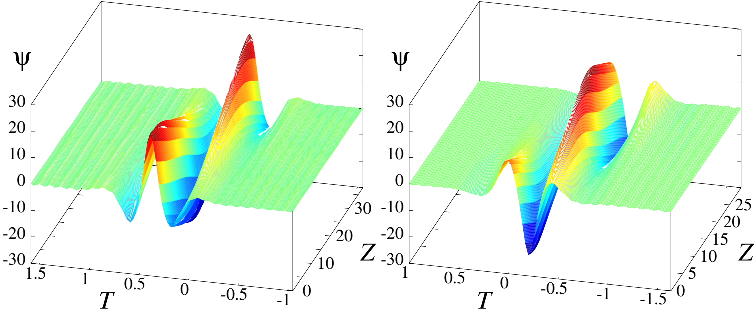
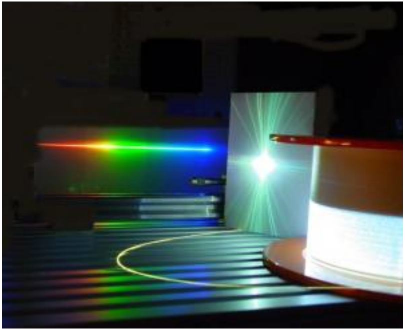


Left: $\nu = 2$, $q = 0.2$; right: $\nu = \sqrt{3}$, $q = 0.4$.

H. Triki, H. Leblond, D. Mihalache, *Phys. Rev. A* **86**, 063825 (2012)

1. Models for few-cycle solitons
   - The mKdV-sG equation
   - General Hamiltonian
2. Supercontinuum generation
   - The phenomenon
   - Towards long wavelengths
   - Self-phase modulation
   - High harmonics generation
3. Few cycle solitons in supercontinuum generation
   - The sG model
   - The mKdV model

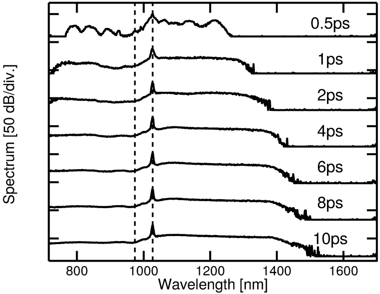
Supercontinuum generation



Supercontinuum generation in PCF. Femto-st lab., Besançon, France

- An intense laser pulse launched in a fiber (photonic crystal fiber especially) turns into white coherent light
- The usual theory uses a generalized NLS model, i.e. slowly varying envelope.
- In principle, SVEA assumes a narrow spectral width!

Supercontinuum generation



M. Andreana *et al.*, *Opt. Express* **20**, 10750 (2012).

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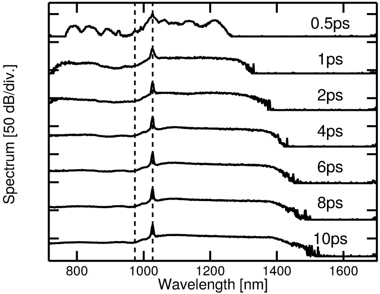

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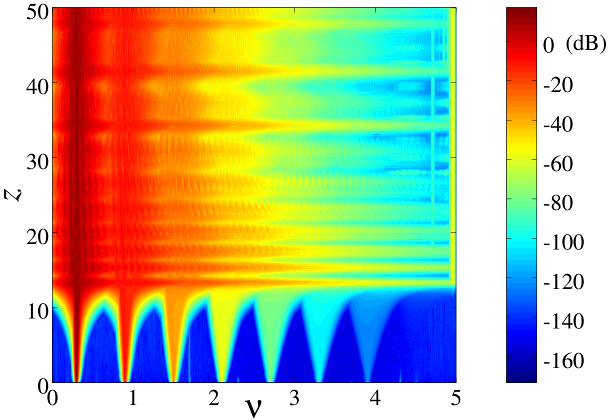
Evidence for supercontinuum generation

- The mKdV equation:
$$\frac{\partial u}{\partial z} + c_2 \frac{\partial u^3}{\partial t} + c_3 \frac{\partial^3 u}{\partial t^3} = 0$$
- Input is a Gaussian pulse:
$$u(0,t) = A \sin(\omega t) e^{-t^2/\tau^2}.$$
- Normalized (dimensionless), $c_2 = c_3 = 1$
- Optical period $\dfrac{2\pi}{\omega}$ and pulse duration τ:
  same as a 100 fs long pulse, $\lambda = 1\mu\mathrm{m}$.
- Numerical resolution........

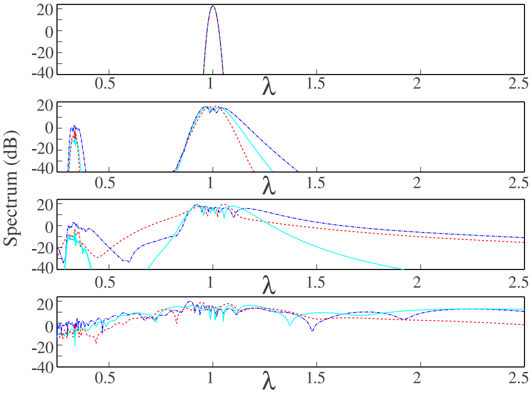


mKdV, input with $FWHM = 100$, $\nu = 0.3$, $A = 0.7$.

- A very broad spectrum is reached quickly.

- Question: generation of frequencies lower than ω?
- We solve numerically the mKdV, sG and mKdV-sG models starting with a 100 fs pulse, $\lambda = 1\mu\mathrm{m}$
  $\longrightarrow$ Compare the evolution of the spectrum

- Comparison between mKdV, sG, and mKdV-sG models



For $z = 0$, 6, 12, and 16.

- sG extends first towards Stokes side,
  mKdV extends first towards anti-Stokes.
- mKdV-sG model lies in between.

- Recall that
  mKdV accounts for UV transitions, and sG for IR transitions.
- Usual Raman broadening yields extension of the spectrum towards low frequencies at the beginning of the process and is due to IR transitions.
- Although Raman effect itself
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- A quasi monochromatic wave

$$u = U(z,t)e^{i(kz-\omega t)} + cc + u_1(z,t), \tag{1}$$

U: fundamental wave amplitude; u_1: small correction.

- Neglect dispersion, and disregard third harmonic generation:

$$U = \frac{A}{2}\exp\left[i\left(\frac{c_1}{2\omega^3} + 3\omega c_2\right)\frac{A^2}{4}z\right]. \tag{2}$$

- Self phase modulation.

  $\longrightarrow$ broadening of the spectrum
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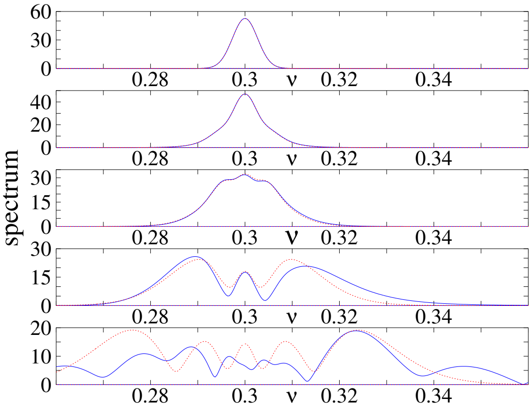
$$U = \frac{A}{2} \exp\left[i\left(\frac{c_1}{2\omega^3} + 3\omega c_2\right)\frac{A^2}{4} z\right]. \tag{2}$$

- Self phase modulation.

  $\longrightarrow$ broadening of the spectrum

  with typical oscillations of the spectral density.

- Numerical mKdV vs analytical self-phase modulation, Eq. (2)



FWHM = 100, $\nu = 0.3$, $A = 0.7$, for $z = 0, 2, 4, 10, 20$.

- Then actual broadening becomes asymmetric,
  while analytic formula remains symmetric.
- The very beginning of the broadening process
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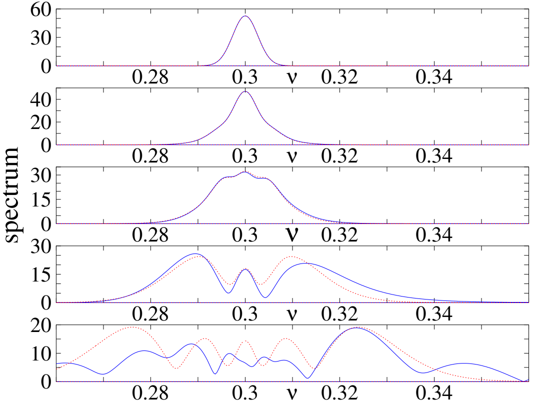

$FWHM = 100$, $\nu = 0.3$, $A = 0.7$, for $z = 0, 2, 4, 10, 20$.

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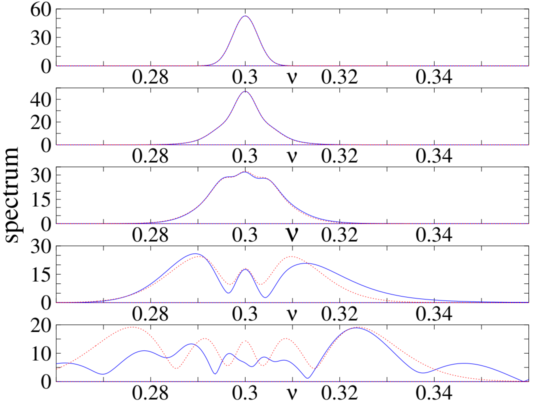

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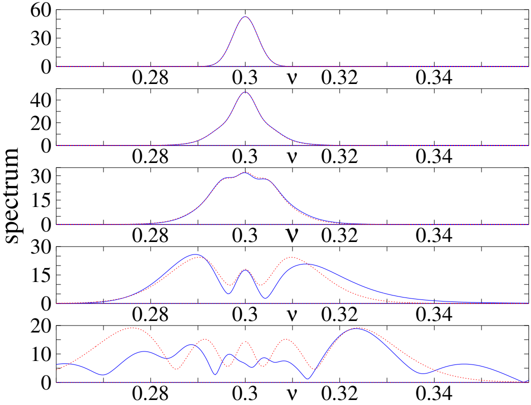

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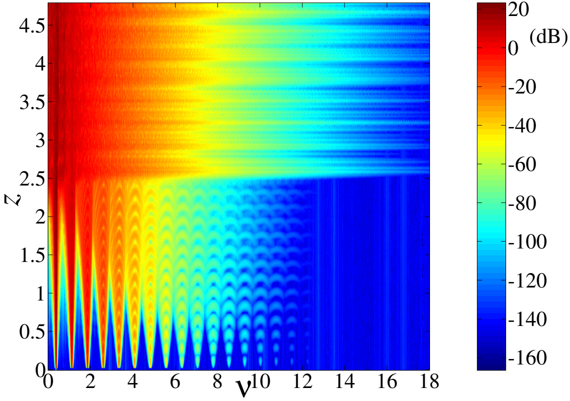
- A lot of high harmonics are created and involved in the process
- The SVEA does not account for high harmonic generation



- Up to 15 harmonics can be seen

  $FWHM = 80$, $\nu = 0.375$, and $A = 5$, sG model.

Spectral width of harmonics

- Assume a Gaussian profile $u = Ae^{\frac{-t^2}{\tau^2}}e^{-i\omega_0 t}$.
- Its Fourier transform is $\hat{u} = \frac{2\sqrt{\pi}}{A\tau}e^{\frac{-(\omega-\omega_0)^2\tau^2}{4}}$.
- The nth harmonic is then $u^n = Ae^{\frac{-nt^2}{\tau^2}}e^{-in\omega_0 t}$,

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- Consequently its Fourier transform is
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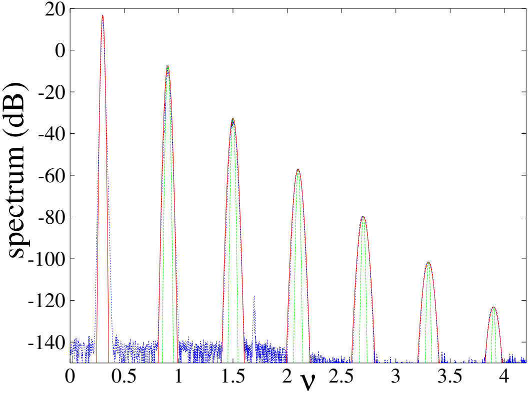
- Spectral widths of the harmonics.
  mKdV compared to Gaussian with width increasing as $\sqrt{n}$



$FWHM = 100$, $\nu = 0.3$, and $A = 0.7$, for $z = 2$.

- Spectral broadening due to parametric interaction between the sidebands of the harmonics and the ones of the fundamental.
- ω_0: fundamental frequency. Assume a sideband $\omega_0 + \delta\omega$,
- the third harmonics contains the sideband $3(\omega_0 + \delta\omega)$.
- It may interact with the fundamental $\pm\omega_0$:
  $\longrightarrow 3(\omega_0 + \delta\omega) - \omega_0 - \omega_0 = \omega_0 + 3\delta\omega$
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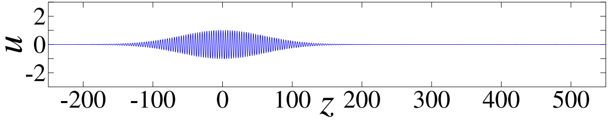
- The sG equation,

$$\frac{\partial u}{\partial z} + c_1 \sin \int^{\tau} u d\tau' = 0$$

$c_1 = 50$, $A = 1$ (a change in c_1 results only in a change of the unit along the z-axis).

- Input is a Gaussian pulse:
  $$u(0,t) = A \sin(\omega t) e^{-t^2/\tau^2}.$$
- A few FCP solitons form and tend to separate



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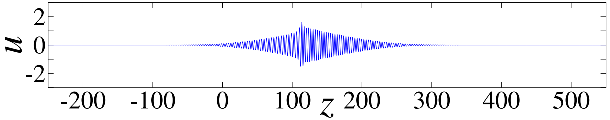

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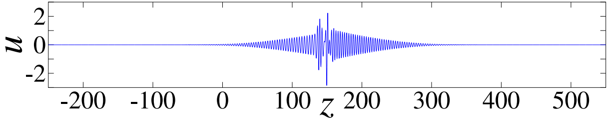

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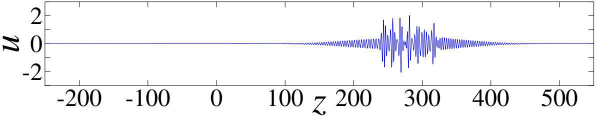
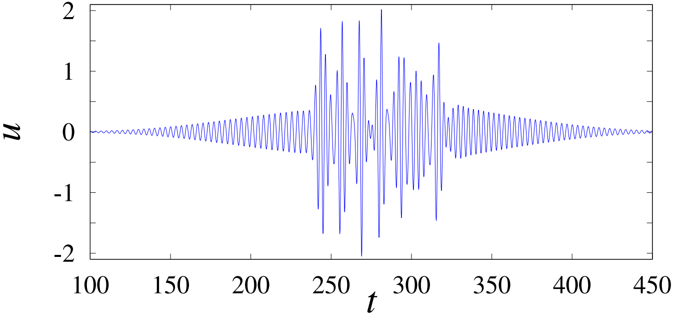


- FCP solitons don't separate completly, but interact
- $A = 1$ a few FCP solitons form and tend to separate



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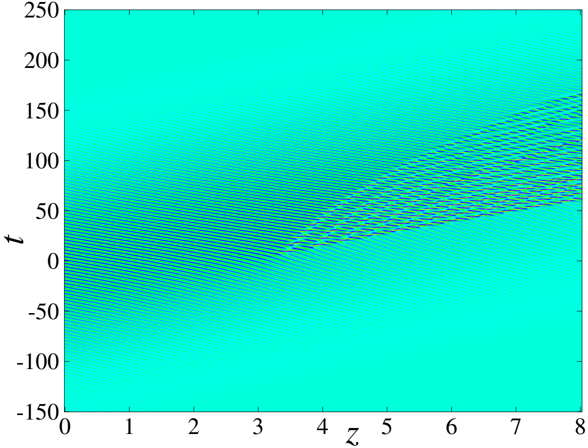
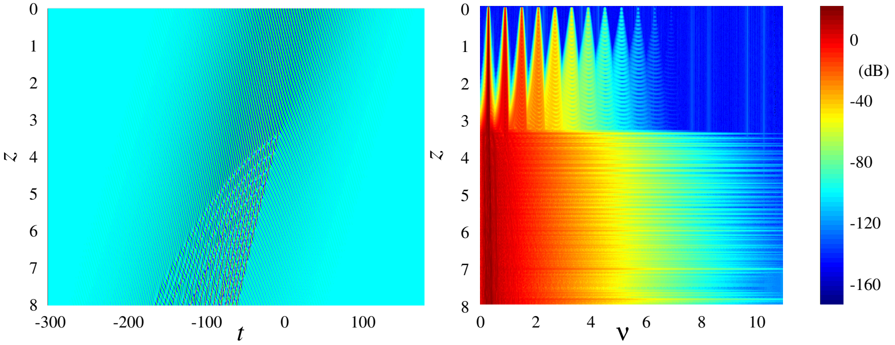


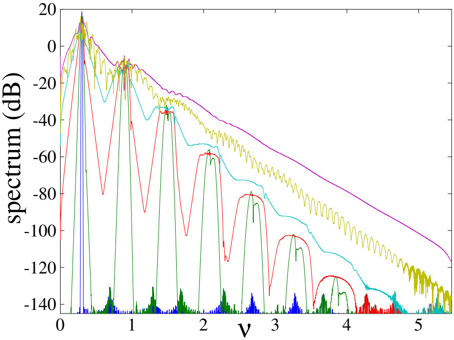
Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 2.5$

- FCP solitons form, don't separate, but interact $\longrightarrow$ ultrabroad supercontinuum



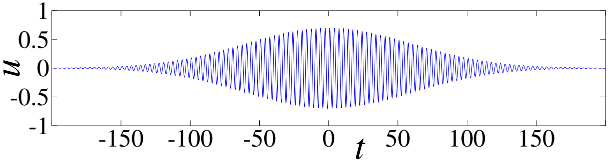
Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 2.5$, according to the sG model.

Evolution of the spectrum according to sG



- Input as above, $z = 0$
- Harmonic generation, $z = 4$
- Broadening of harmonics, $z = 8$
- Arising of the first soliton, $z = 9.2$
- The second soliton is just formed, $z = 11.16$
- Final ultrabroad spectrum, $z = 20$

- At high amplitude level ($A \sim 2$) an initial 100 fs pulse splits into a set of FCP solitons, which are mKdV breathers



- The FCPs form in a first stage until $z \simeq 12$ then slowly go away one from the other.
- A huge spectral broadening occurs when they form.

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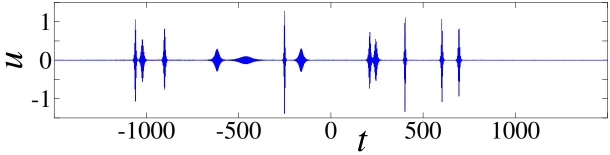

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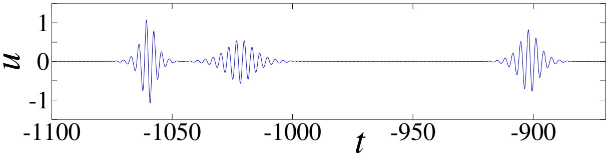

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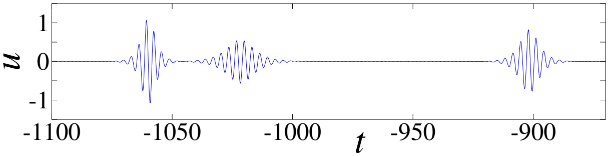

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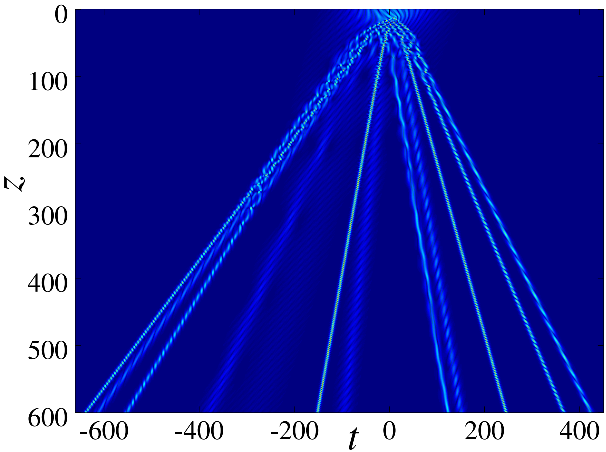
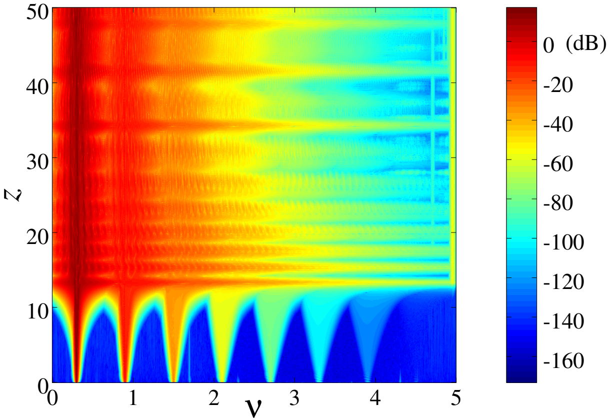


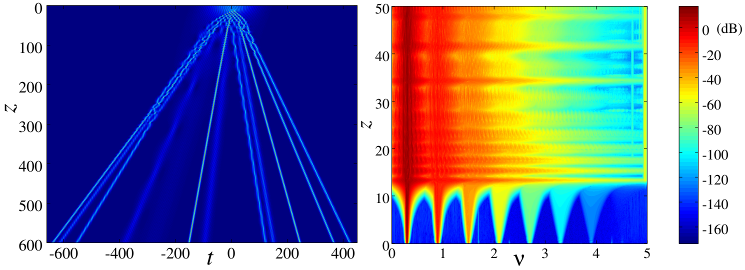
Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 0.7$, according to mKdV.

- The input pulse splits into a set of FCP solitons



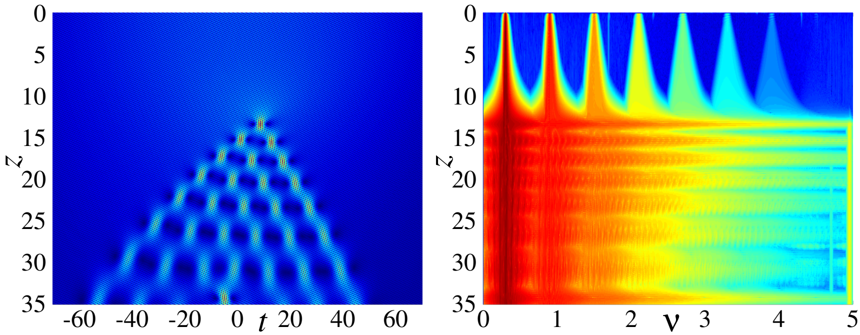
mKdV, input with $FWHM = 100$, $\nu = 0.3$, $A = 0.7$.

- A very broad spectrum is reached quickly.



mKdV, input with $FWHM = 100$, $\nu = 0.3$, $A = 0.7$.

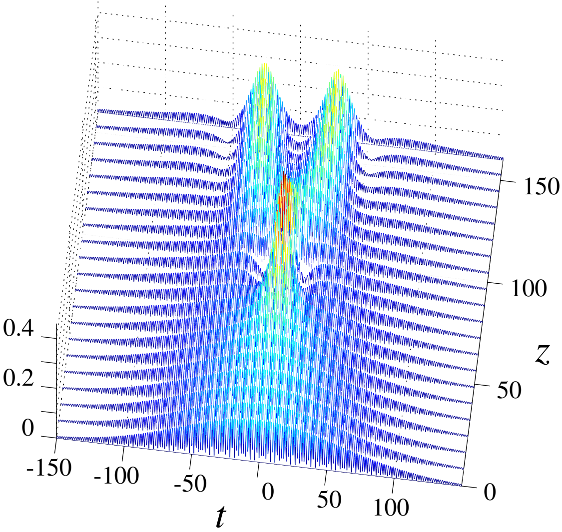
- Compare temporal profile and spectrum



mKdV, input with $FWHM = 100$, $\nu = 0.3$, $A = 0.7$.

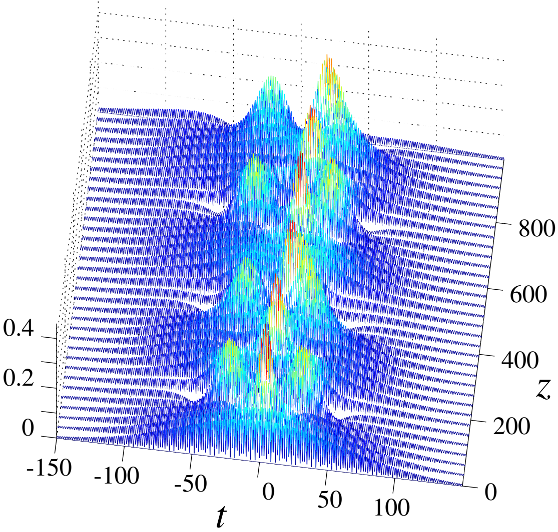
- The spectrum extends when the solitons separate

- At a lower amplitude ($A \sim 0.1, 0.2$), it may happen, that two solitons only form



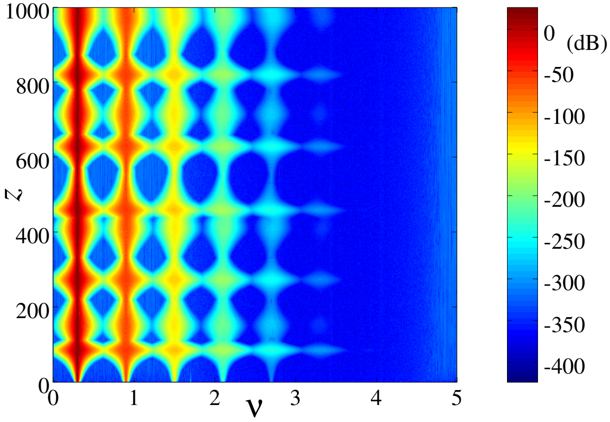
Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 0.17$.

- At a lower amplitude ($A \sim 0.1, 0.2$),
  they interact with each other and form a kind of superbreather



Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 0.17$.

- Then the spectral width oscillates



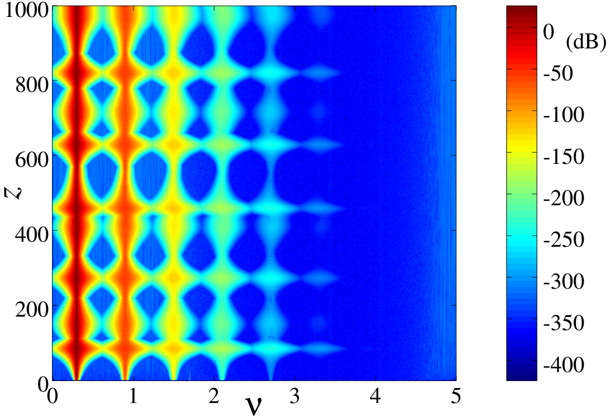
Initial pulse with $FWHM = 100$, $\nu = 0.3$, and $A = 0.17$.

Conclusion

- Generic non-slowly varying envelope approximation models: modified Korteweg-de Vries, sine-Gordon, mKdV-sG
- Adequately describe the supercontinuum generation over several octaves
- sG accounts for the Raman broadening
- Modulation instability
- High harmonics generation
- FCP soliton generation and interaction leads to ultrabroad spectrum.

H. Leblond, Ph. Grelu, Dumitru Mihalache, Models for supercontinuum generation beyond the slowly-varying-envelope approximation, *Phys. Rev. A* **90**, 053816 (2014)

Thank you for your attention.



1. Models for few-cycle solitons
   - The mKdV-sG equation
   - General Hamiltonian

2. Supercontinuum generation
   - The phenomenon
   - Towards long wavelengths
   - Self-phase modulation
   - High harmonics generation

3. Few cycle solitons in supercontinuum generation
   - The sG model
   - The mKdV model