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Motion of solitons in CGL-type equations

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Galilean invariance of CGL

- The CGL equation is

$$\frac{\partial E}{\partial z} = \delta E + \left(\beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) E |E|^2 + (\mu + i\nu) E |E|^4$$

- If $\beta = 0$, and $E_0(z, t)$ solution to CGL

$$E = E_0(z, t - wz) \exp i [Dwt - (Dw^2/2) z]$$

is a solution moving at inverse speed w .

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E : electric field amplitude;

δ : net linear gain;

β : spectral gain bandwidth;

$D = \pm 1$: dispersion;

ε : cubic nonlinear gain;

μ : quintic nonlinear gain;

ν : 4th-order nonlinear index;

z : number of round-trips;

- If $\beta = 0$, and $E_0(z, t)$ solution to CGL

$$E = E_0(z, t - wz) \exp i \left[D\omega t - \left(D\omega^2/2 \right) z \right]$$

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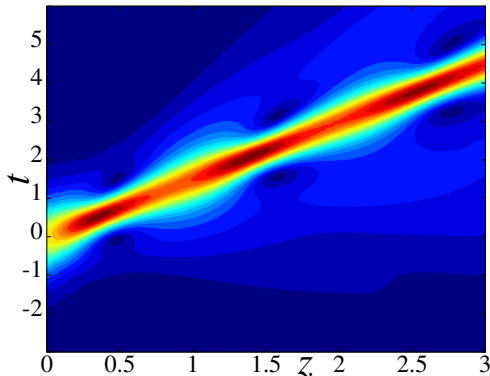
is a solution moving at inverse speed w .

Galilean invariance of CGL

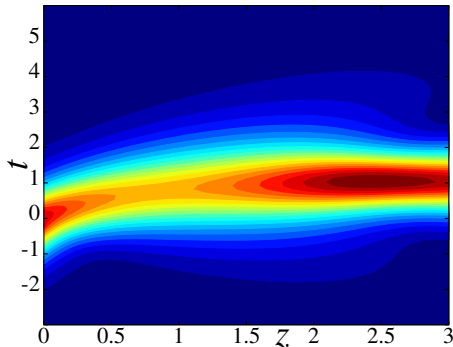
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$$E = E_0(z, t - wz) \exp i [Dwt - (Dw^2/2) z]$$

is a solution moving at inverse speed w .



- $\beta \partial^2 E / \partial t^2$ breaks Galilean invariance and prevents any motion of the solitons.
- If $\beta \neq 0$: the moving soliton does not exist.



At $z = 0$, $E = E_0 \exp i\Delta\omega t$, $\beta = 0.55$ instead of 0

- Strong breaking due to the limited gain bandwidth.

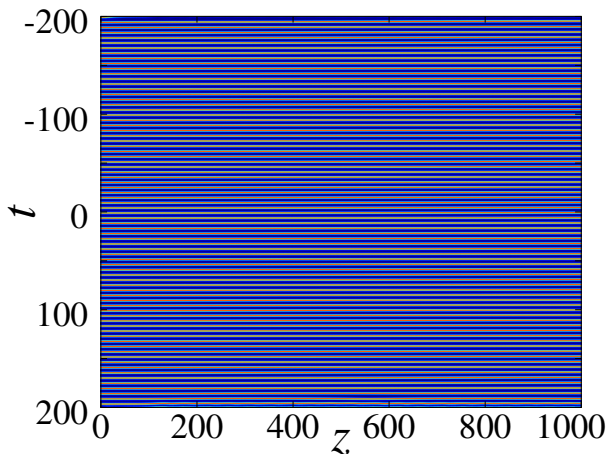
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- Experiments in mode-locked fiber lasers:
 \implies existence of “soliton gas”
- A large number of solitons in motion.
- Spectral bandwidth of gain is finite: $\beta \neq 0$
- Is the motion it due to the cw component?
- Try to inject cw.

$$\frac{\partial E}{\partial z} = \delta E + \left(\beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) E |E|^2 + (\mu + i\nu) E |E|^4 + A \exp(-i\Delta\omega_0 t)$$

A: amplitude of injected cw; $\Delta\omega_0$: frequency shift.

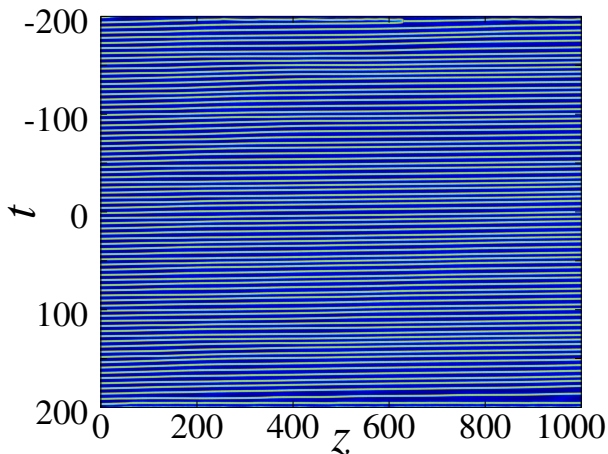
Changing “states of matter”: soliton crystal



- Soliton crystal.

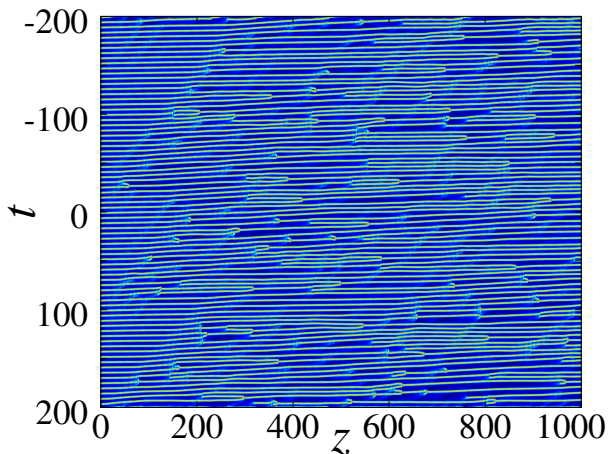
$$\Delta\nu_0 = 1.2.$$

Changing “states of matter”: soliton liquid



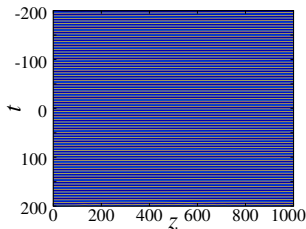
- Soliton liquid.
 $\Delta\nu_0 = 0.9.$

Changing “states of matter”: soliton gas

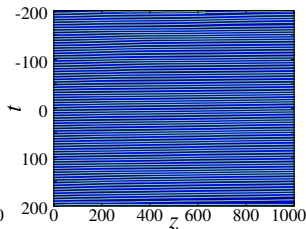


- Soliton gas.
 $\Delta\nu_0 = 0.8.$

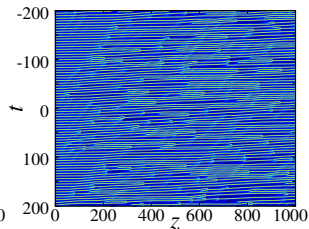
Changing “states of matter” of solitons



Soliton crystal,
 $\Delta\nu_0 = 1.2$,

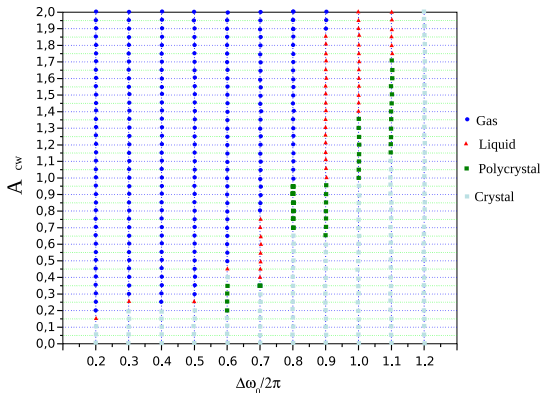
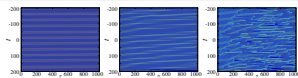


Soliton liquid,
 $\Delta\nu_0 = 0.9$,



Soliton gas.
 $\Delta\nu_0 = 0.8$.

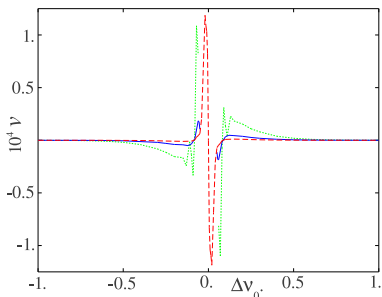
Changing “states of matter” of solitons



- The different regimes vs detuning $\Delta\omega_0$ and amplitude $A = A_{cw}$ of injected cw

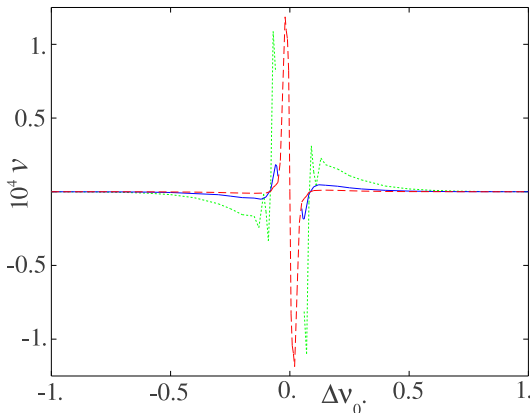
- For a single soliton, input of the form $E = E_0 \exp i\Delta\omega_1 t$, varying A , $\Delta\omega_0$ and $\Delta\omega_1 \Rightarrow$ Pulse motion.

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- The velocity depends on A and $\Delta\omega_0$ (injected cw), but not on $\Delta\omega_1$ (initial speed).
i.e. **Speed is entirely determined by injected cw.**



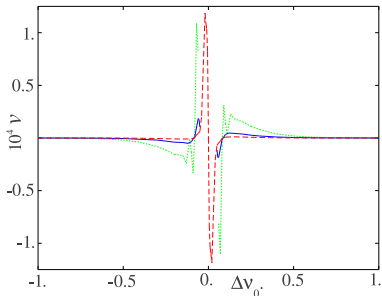
Soliton velocity vs $\Delta\nu_0 = \Delta\omega_0/2\pi$ for $A = 0.004$ (green dotted),
0.002 (red dashed), 0.001 (blue solid line).

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Soliton velocity vs $\Delta\nu_0 = \Delta\omega_0/2\pi$ for $A = 0.004$ (green dotted), 0.002 (red dashed), 0.001 (blue solid line).

- Motion is restored but Galilean invariance is not.

“Brownian motion” induced by injected cw

- Soliton velocity is fixed by the cw.
- At higher amplitudes of injected cw:
More complex nonlinear interaction between cw and solitons.
⇒ The amplitude of cw (radiation) varies with t .
- A tiny variation of the cw component in either amplitude or frequency changes radically the soliton velocity
⇒ apparently random variations of the the soliton speed.
- ⇒ Erratic motion of solitons,
and soliton gas.

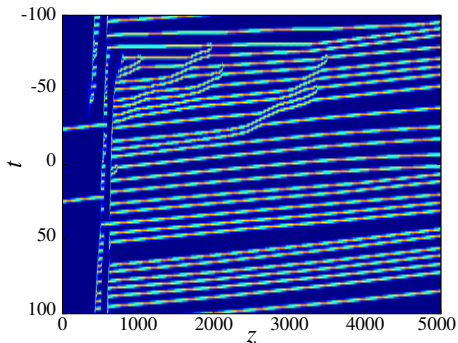
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- An integral term accounting for the fast gain dynamics

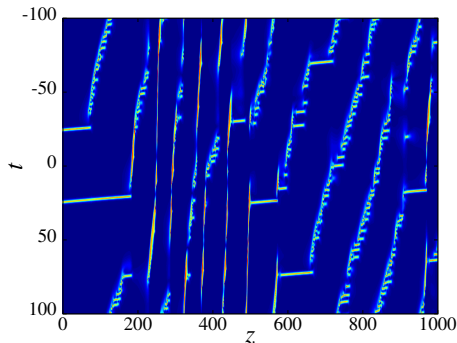
$$\frac{\partial E}{\partial z} = \delta E + \left(\beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) E |E|^2 + (\mu + i\nu) E |E|^4 - \Gamma E \int_{-\infty}^t (|E|^2 - \langle |E|^2 \rangle) dt'$$

- Represents the decrease of the population inversion (and hence of gain) when stimulated emission occurs.

- A single pulse (or two-pulse) input is unstable:
New pulses form in front of the input (towards $t < 0$), and quickly disappear.
- Unstable pulse emission repeats all along the cavity,
⇒ multi-pulse pattern
- Solitons move slowly



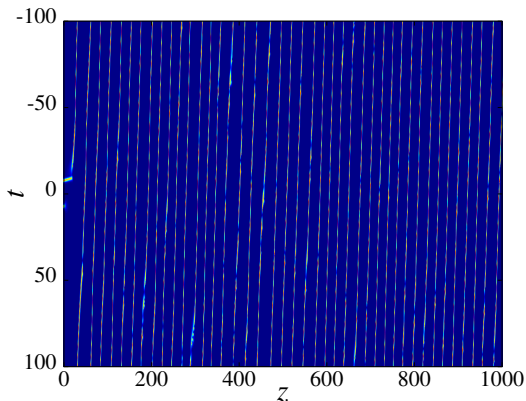
- For larger Γ , the instability increases: pulses form and vanish faster.
- Then, the pulse train does not stabilize any more: pulses are created and vanish permanently



$$\Gamma = 0.03$$

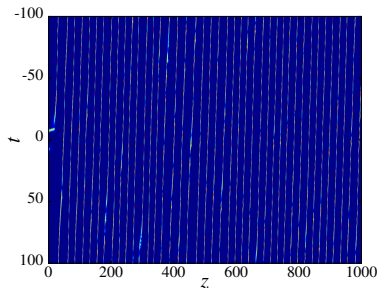
- Generation and vanishing process \implies effective soliton motion

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- The inverse velocity w of this motion is very large



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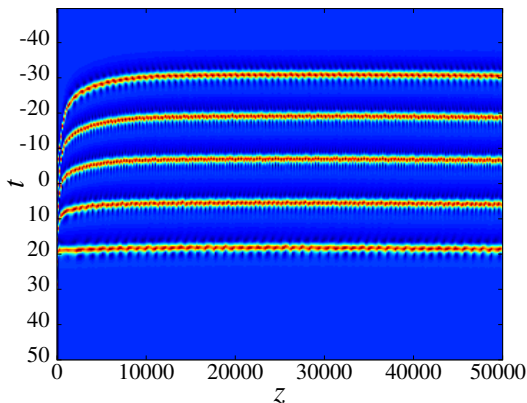
- For high values of Γ , the moving soliton is unstable and vanishes.
- Gain dynamics can induce soliton motion.

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- Model including external injection, fast gain dynamics, and gain saturation.

$$\begin{aligned} \frac{\partial E}{\partial z} = & \left(\frac{g_0}{1 + \langle |E|^2 \rangle / I_s} - r \right) E + \left(\beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} \\ & + (\varepsilon + i) E |E|^2 + (\mu + i\nu) E |E|^4 \\ & - \Gamma E \int_{-\infty}^t (|E|^2 - \langle |E|^2 \rangle) dt' + A \exp(-i\Delta\omega_0 t) \end{aligned}$$

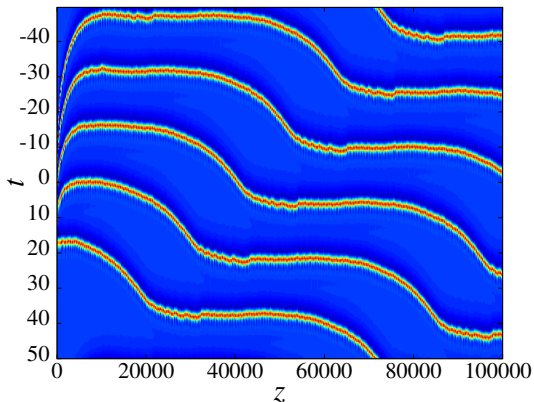
- Gain saturation limits the number of solitons
 ⇒ A liquid: condensed phase, which does not fill the box



$\Delta\nu_0 = 0.1$, $A = 0.115$, with velocity compensation, $w = -0.05877$.

- Not a crystal: no phase-locking

- We can still have a soliton gas:

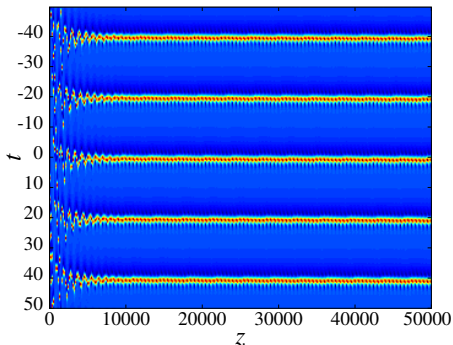


$$\Delta\nu_0 = 0.1, A = 0.120, w = -0.07040.$$

Harmonic mode-locking

- Equidistant solitons filling all the box, stable state.

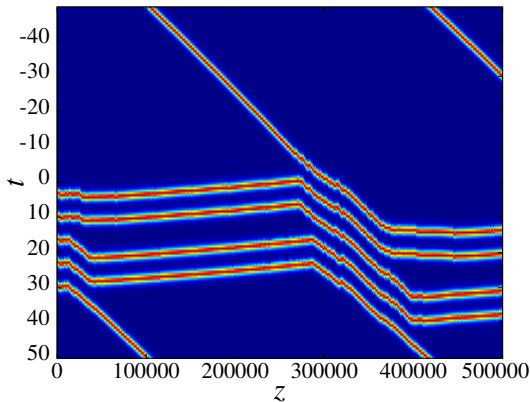
When $0.125 \leq A \leq 0.133$.



$$\Delta\nu_0 = 0.1, A = 0.130, w = -0.02755.$$

- Consecutive pulses are phase-locked: a crystal, but the crystal length exactly matches the box length.

- In a three bunch pattern,
elastic interaction according to the Newton's cradle scenario



$$A = 0.130, \Delta\nu_0 = 0.5 \text{ and } w = -0.0024.$$

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- The CGL equation

 $(u = E)$

$$u_z = \delta u + \left(\beta + i \frac{D}{2} \right) u_{tt} + (\varepsilon + i) u |u|^2 + (\mu + i\nu) u |u|^4.$$

- Moving solution: $u = u_0(t - T, z) e^{i(\omega t - kz)}$
with $u_0(t, z)$ solution to CGL with $\beta = 0$,
 $T = Vz$, $\omega = \frac{V}{D}$ and $k = \frac{V^2}{2D}$.
- Perturbative approach: Consider some small non zero β .
 u a soliton solution,

$$M = \int_{-\infty}^{+\infty} |u|^2 dt : \text{its mass,}$$

$$T = \int_{-\infty}^{+\infty} t |u|^2 dt / M : \text{position of its center of mass.}$$

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- The velocity of the pulse is then

$$\frac{dT}{dz} = \frac{1}{M} \int t (u_z u^* + cc) dt$$

(cc: complex conjugate, $u_z = \partial u / \partial z$).

- Using the CGL equation:

$$\frac{dT}{dz} = \frac{1}{M} \left(I_1 + \frac{iD}{2} I_2 + \beta I_3 \right) dt,$$

where

$$I_1 = 2 \int t (\delta |u|^2 + \varepsilon |u|^4 + \mu |u|^6) dt,$$

$$I_2 = \int t (u_{tt} u^* - cc) dt,$$

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- Assumption: u_0 is a symmetrical pulse, centered at $t = T$, consequently the function $u_0(t')$, with $t' = t - T$, is even.

- Then

$$I_1 = 2 \int t (\delta |u|^2 + \varepsilon |u|^4 + \mu |u|^6) dt,$$

becomes

$$I_1 = 2T \int (\delta |u_0|^2 + \varepsilon |u_0|^4 + \mu |u_0|^6) dt'.$$

and so on.

- Then we can compute the acceleration $d^2 T / dz^2$:

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- Finally; we obtain the expression of the force $F = Md^2 T/dz^2$:

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$$M \frac{dV}{dz} = F = -4\beta \int |u_{0t'}|^2 dt' V,$$

- Hence, the velocity evolves as $V(z) = V(0)e^{-\lambda z}$ with the decay rate

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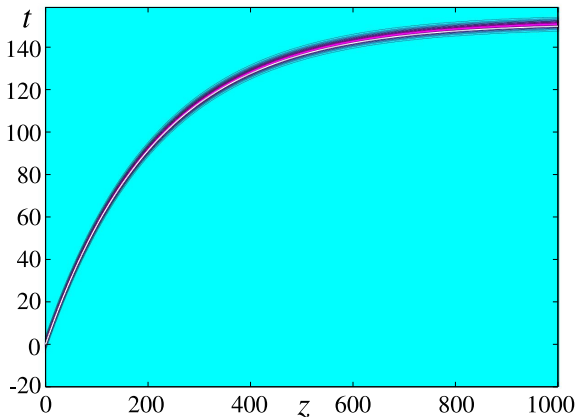
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- An example of calculation

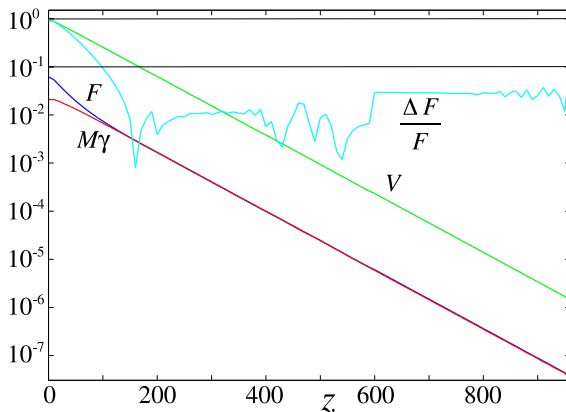


Initial velocity $V_0 = 0.7$ and gain bandwidth coefficient $\beta = 0.004$.

White line: approximate analytical solution

- Good agreement with the numerical solution.

- We plot the characteristics of the pulse motion vs z



Logarithmic scale.

V : velocity; γ : acceleration and M : mass; F : force from above theory;
 $\Delta F/F$: relative difference between F and $M\gamma$.

Parameters: $\omega = 1$, $\beta = 0.0124$.

- Effect of finite bandwidth of gain on CGL soliton with anomalous dispersion is equivalent to a **viscous friction force**, if it is not too large.
- To construct simplified models to describe CGL soliton interactions as **forces between effective particles** :
- The **lack of Galilean invariance** of CGL was a major difficulty, since the concept of force is based on it.
- With our result, we can approach soliton interaction in a **conservative frame**.
Then, the finite bandwidth of gain could be treated as a phenomenological friction force.

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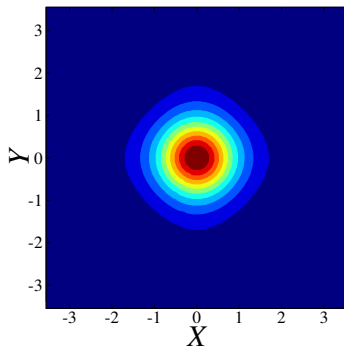
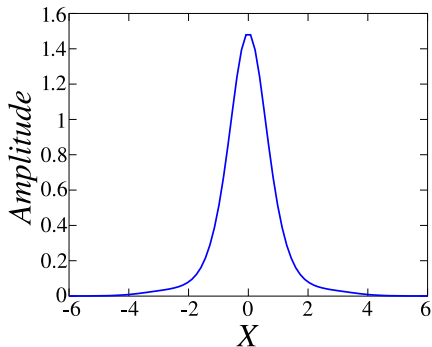
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- The **lack of Galilean invariance** of CGL was a major difficulty, since the concept of force is based on it.
- With our result, we can approach soliton interaction in a **conservative frame**.
Then, the finite bandwidth of gain could be treated as a phenomenological friction force.

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 - **Fondamental soliton**
 - Dipoles and vortices

- (2+1)-D spatial Ginzburg-Landau equation:

$$\frac{\partial u}{\partial Z} = \left[-\delta + iV(X, Y) + \frac{i}{2} \nabla_{\perp}^2 + (i + \epsilon)|u|^2 - (i\nu + \mu)|u|^4 \right] u,$$

- $\nabla_{\perp}^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$: the paraxial diffraction
- A periodic potential: $V(X, Y) = -V_0 [\cos(2X) + \cos(2Y)]$ breaks Galilean invariance
- $\delta = 0.4, \epsilon = 1.85, \mu = 1, \nu = 0.1, V_0 = 1$,
for which the quiescent fundamental soliton is stable.

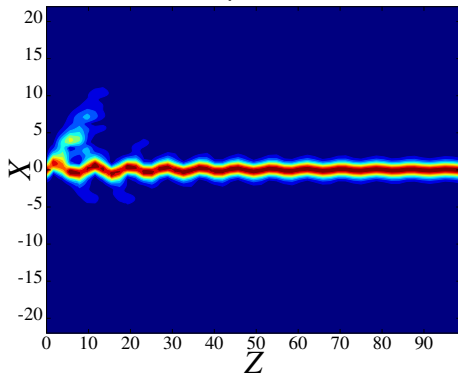

 $|u(X, Y)|;$

 $|u(X)|$ at $Y = 0$.

The stable fundamental soliton

- Input $u = u_0 \exp(i\mathbf{k}_0 \mathbf{R})$,
with $\mathbf{R} = (X, Y)$, and $\mathbf{k}_0 = (k_0 \cos \theta, k_0 \sin \theta)$ ($0 \leq \theta \leq \pi/4$)
- In an amplifier, the factor $(i\mathbf{k}_0 \mathbf{R})$
represents a deviation of the wave vector \mathbf{k} from the Z -axis.
- Indeed, CGL is derived within the SVEA:
Either $E = \mathcal{U}(X, Y, Z - vT) e^{i(k_X X + k_Y Y + k_Z Z - \omega T)} + \text{c.c.}$,
or $E = u(X, Y, Z - vT) e^{i(k_Z Z - \omega T)} + \text{c.c.}$,
with $u(X, Y, Z - vT) = \mathcal{U}(X, Y, Z - vT) e^{i(k_X X + k_Y Y)}$.
Equivalent if k_X, k_Y are small enough.

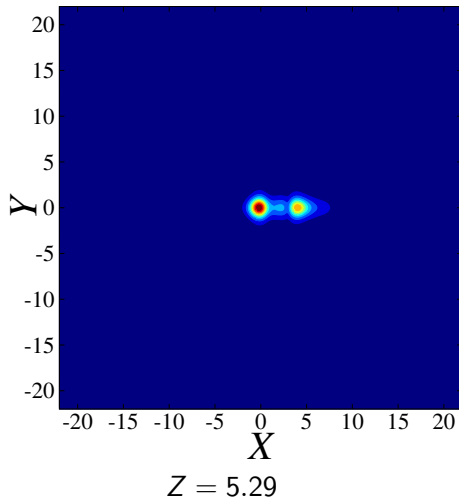
Fundamental soliton with initial kick

- If k_0 is small, the pulse oscillates in the potential site

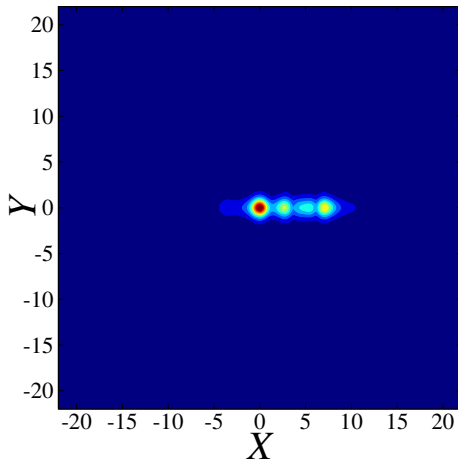


$|u(X,Z)|$ in the cross section $Y = 0$, for $k_0 = 1.61$, $\theta = 0$.

- For larger k_0 , the pulse starts to move

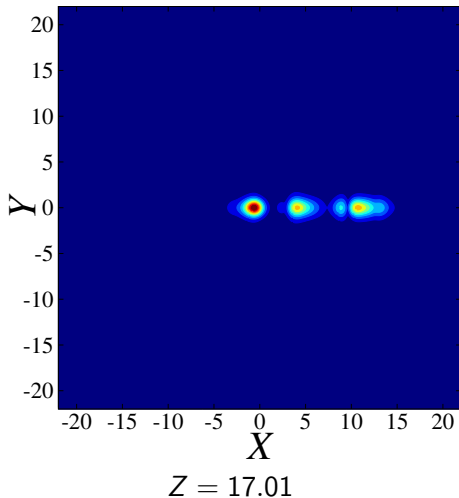


- For larger k_0 , the pulse starts to move

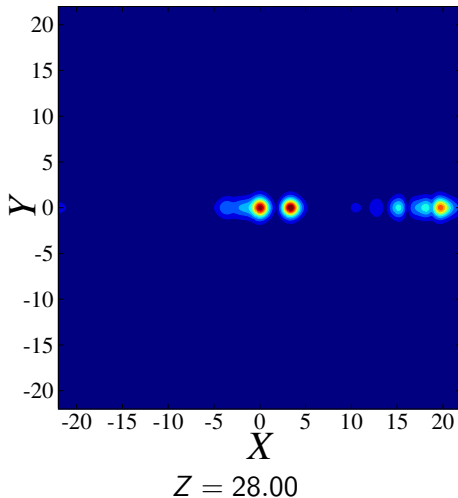


$Z = 9.39$

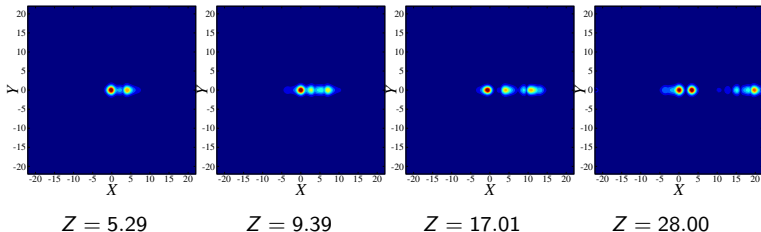
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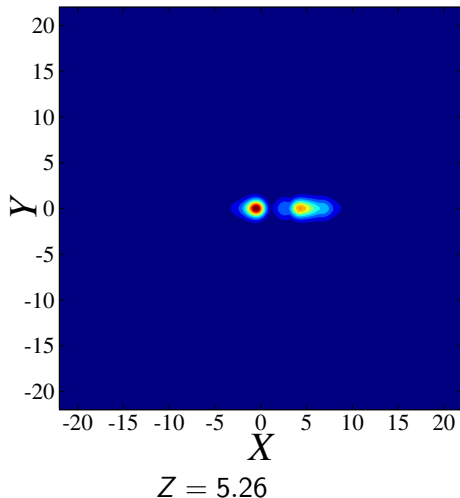
- For larger k_0 , the pulse starts to move



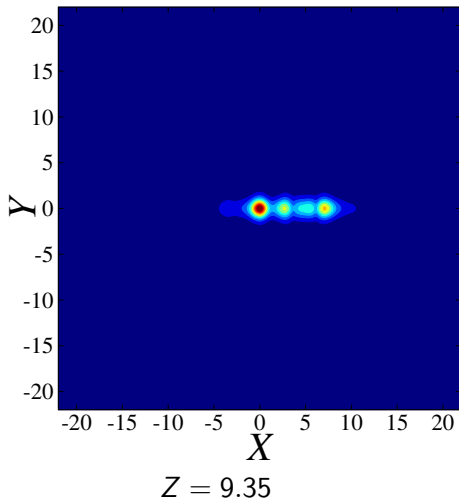
$|u(X,Z)|$ at $Y = 0$, for $k_0 = 1.6878$, $\theta = 0$.

- The pulse leaves a copy of it behind it

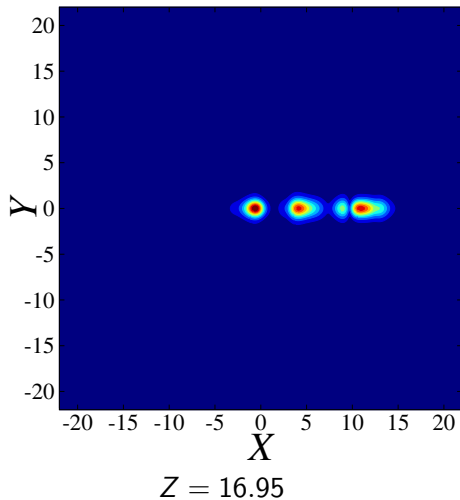
- Another example, increasing k_0 :



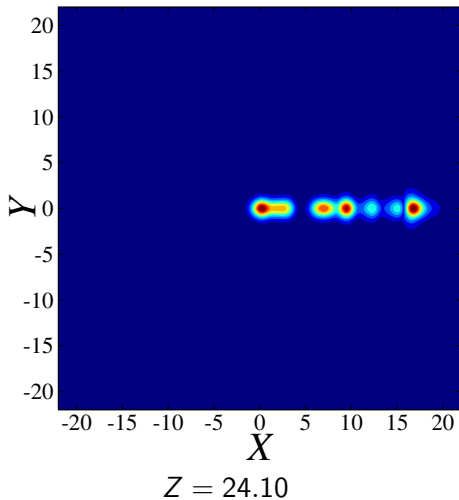
- Another example, increasing k_0 :



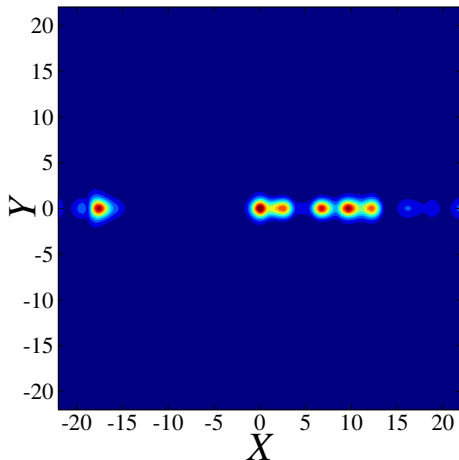
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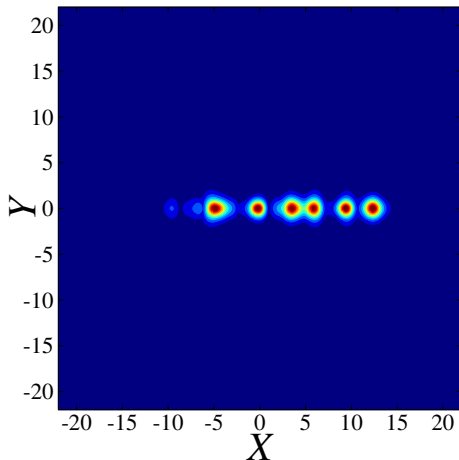


- Another example, increasing k_0 :



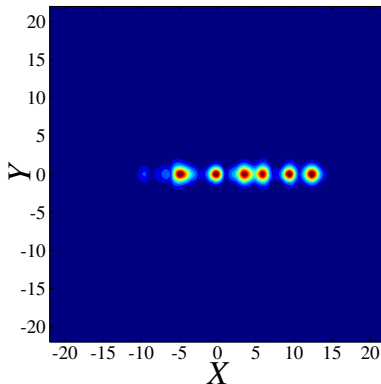
$$Z = 35.33$$

- Another example, increasing k_0 :



$$Z = 48.24.77$$

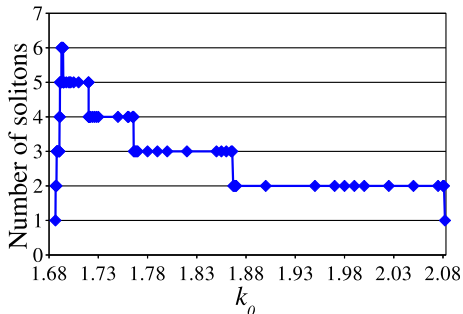
- Another example, increasing k_0 :



$$Z = 48.24.77, \text{ for } k_0 = 1.694$$

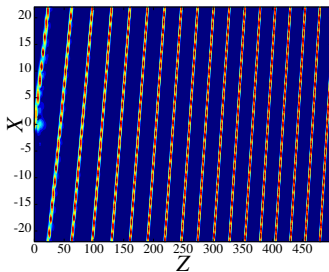
- An arrayed set of 5 fix + 1 moving solitons.

- The total number of emitted solitons first grows fast with k_0 .

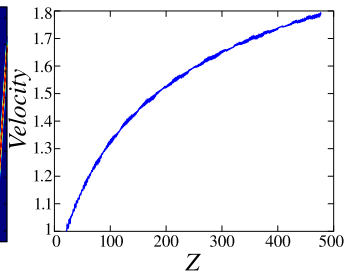


- It reaches a maximum of 5 (6 with the initial one)
- Then slowly goes down to 0 (the initial one only)

- For the highest k_0 , the soliton moves freely



$|u(X,Z)|$ at $Y = 0$,

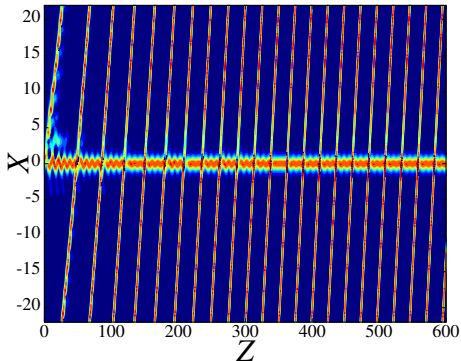


transverse speed, $k_0 = 2.1$, $\theta = 0$.

- The soliton velocity increases, approaching a certain limit value

Periodic elastic collisions

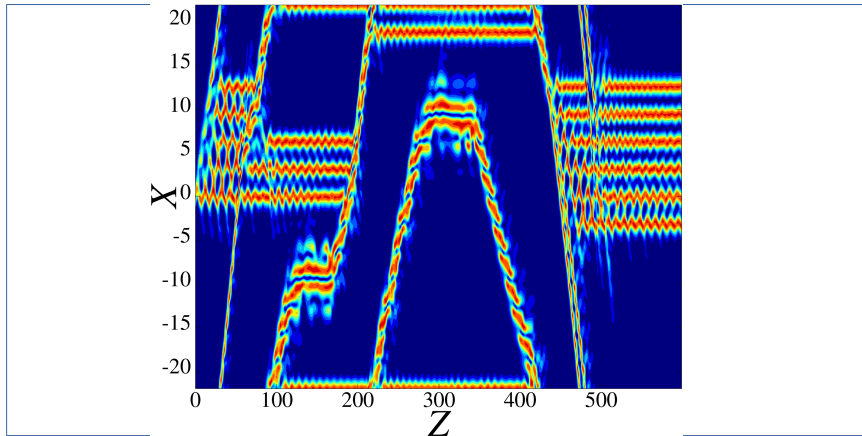
- A moving soliton with one fix soliton



$$k_0 = 1.867, \theta = 0.$$

- An example of the Newton's-cradle scenario

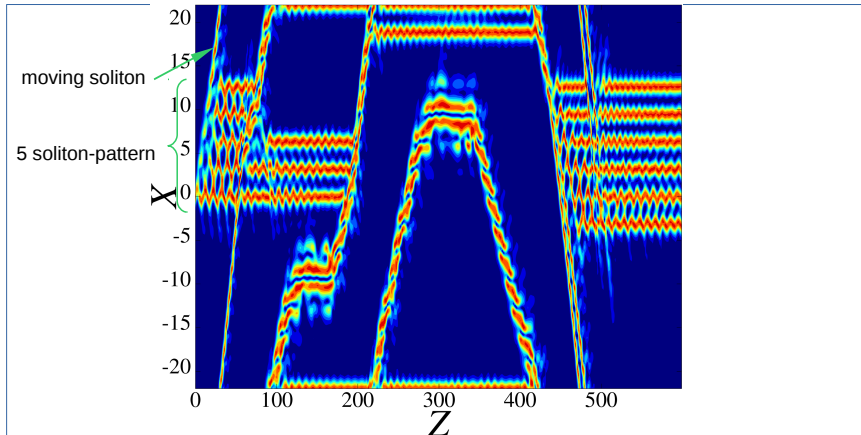
- 1 moving and 5 fix solitons



$|u(X,Z)|$ at $Y = 0$; $k_0 = 1.693$, $\theta = 0$.

- An quite complex interaction

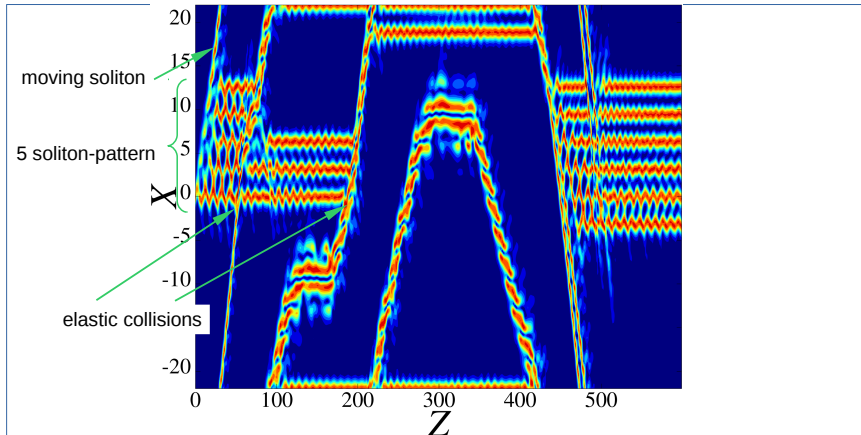
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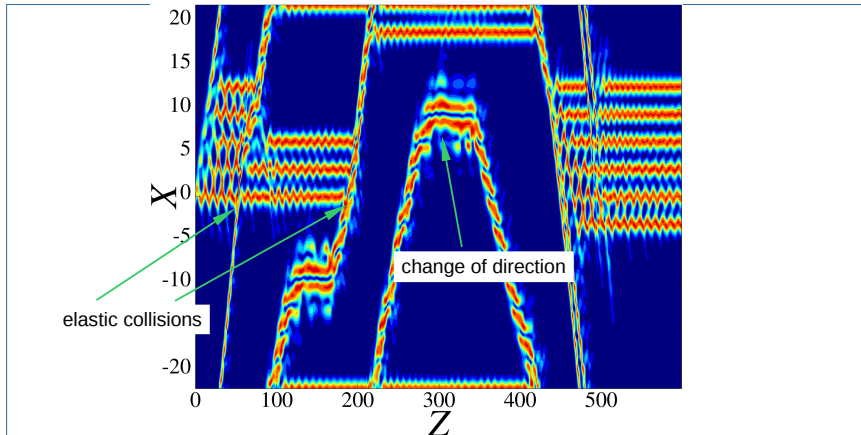
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$|u(X,Z)|$ at $Y = 0$; $k_0 = 1.693$, $\theta = 0$.

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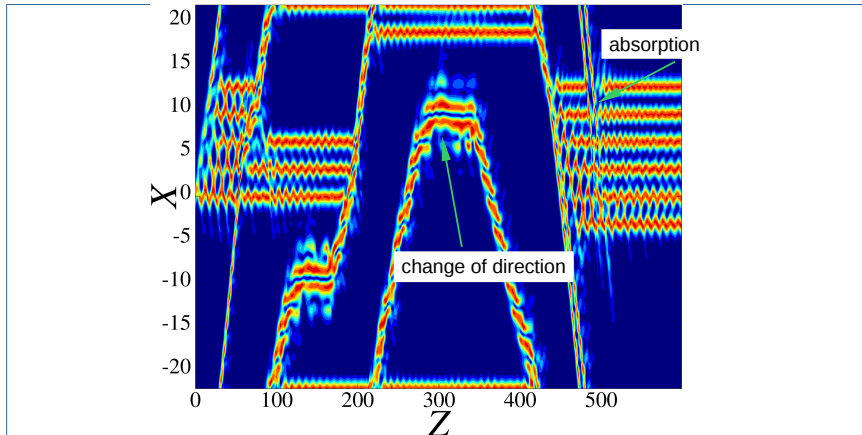
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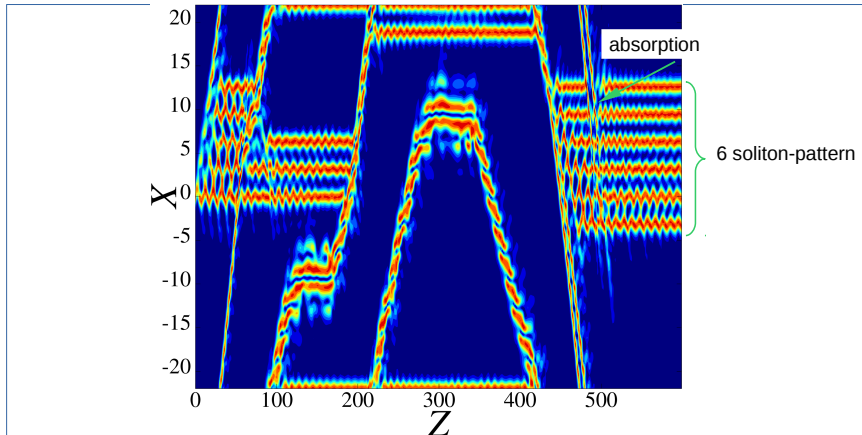
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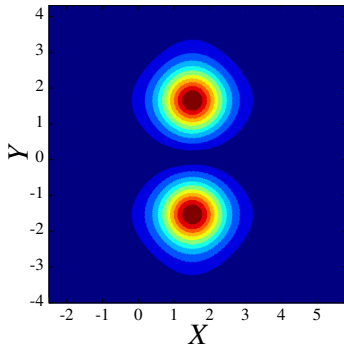


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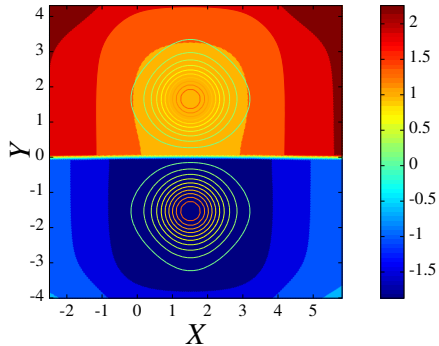
- An quite complex interaction

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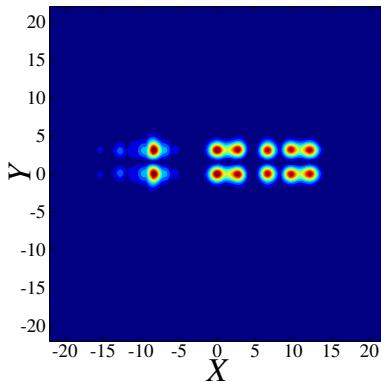
The dipole



Amplitude

phase (in units of π)

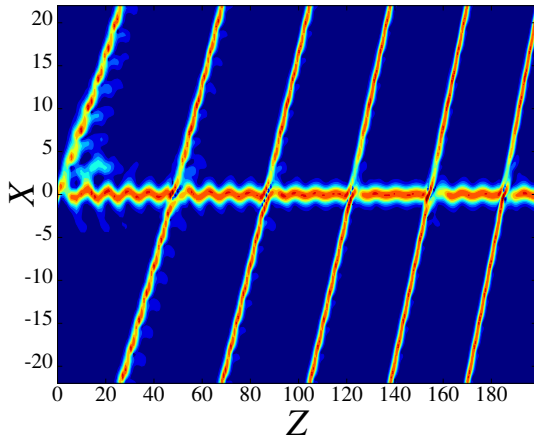
- Moving the dipole



$|u(X, Y)|$, at $Z = 22.410$, for $k_0 = 1.665$, $\theta = 0$.

- 5 fix and 1 moving dipoles

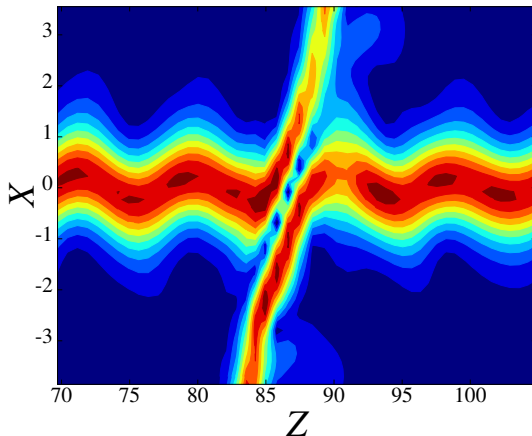
- Interaction of 1 moving dipole with 1 fix dipole



$|u(X, Y, Z)|$ at $Y = 0$, for $k_0 = 1.865$.

- Repeated elastic collisions
- An example of the Newton's cradle scenario

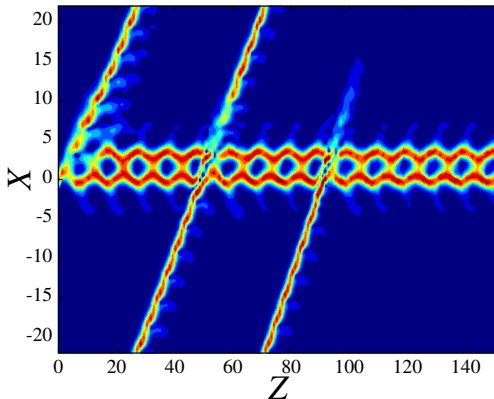
- Interaction of 1 moving dipole with 1 fix dipole



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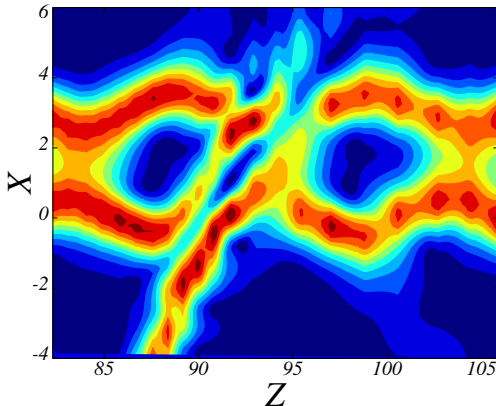
- The Newton's cradle with absorption scenario



$|u(X, Y, Z)|$ at $Y = 0$, for $k_0 = 1.816$.

- After several quasi-elastic elastic collisions, the moving dipole is eventually absorbed by the quiescent
- Interaction of 1 moving dipole with 2 fix dipoles

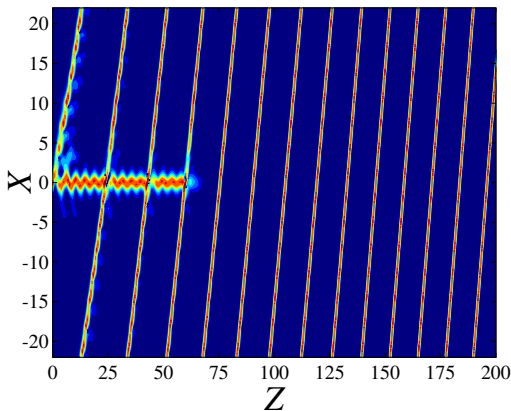
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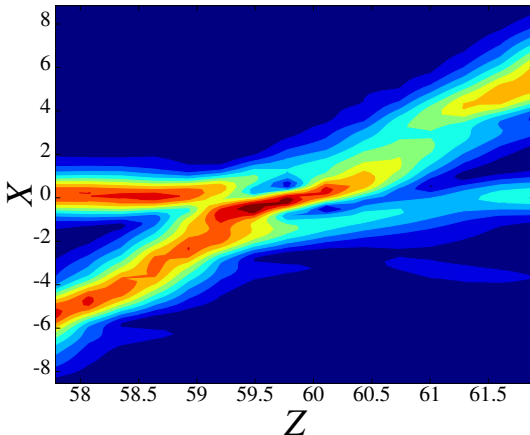
- Transient Newton's cradle with clearing the obstacle



$|u(X, Y, Z)|$ at $Y = 0$, for $k_0 = 1.884$.

- After several quasi-elastic elastic collisions, the moving dipole absorbs the stationary chain

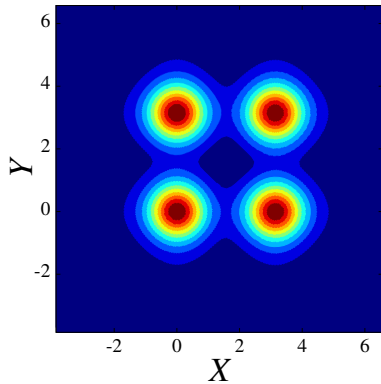
- Transient Newton's cradle with clearing the obstacle



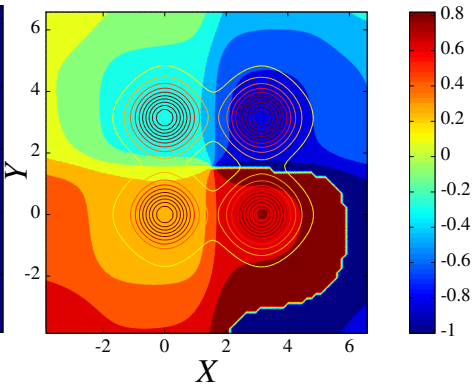
$|u(X, Y, Z)|$ at $Y = 0$, for $k_0 = 1.884$.

- After several quasi-elastic elastic collisions, the moving dipole absorbs the stationary chain

Square-shaped (offsite-centered) vortex.



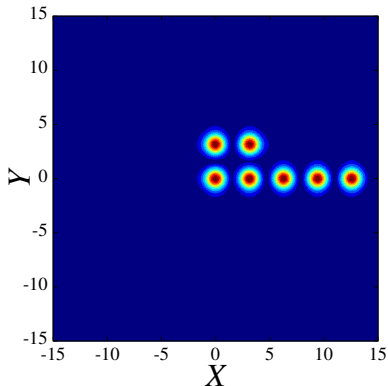
Amplitude



phase.

- It is unstable

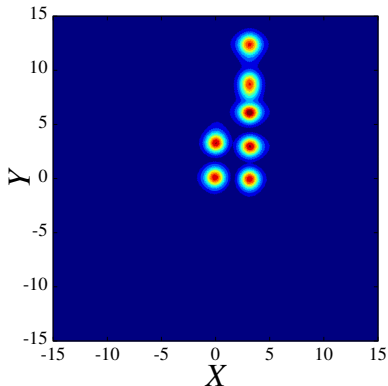
- Moving the vortex



Amplitude at $Z \simeq 300$, for $k_0 = 1.5$, and $\theta = \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$.

- A set of fundamental solitons is formed
- For a clockwise rotating vortex, solitons form on the **other line**

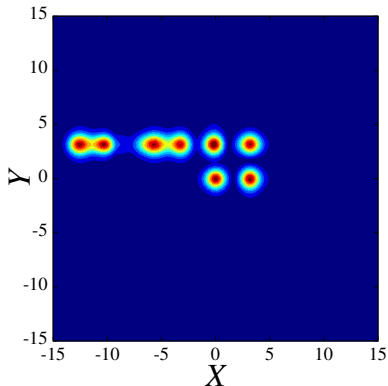
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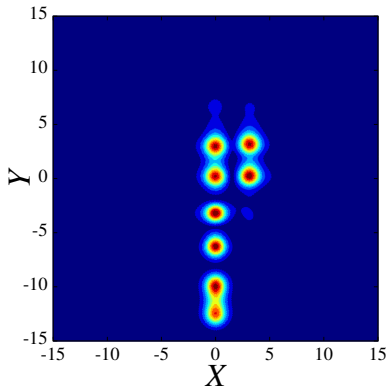
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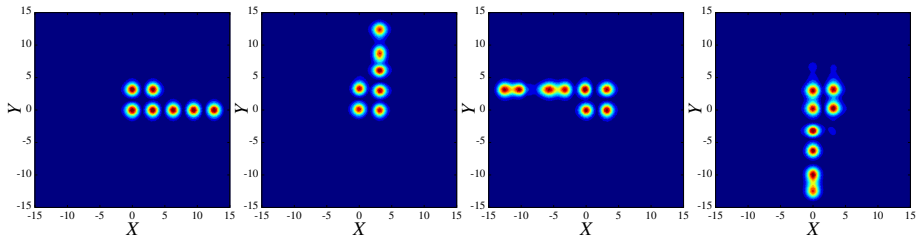
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- A set of fundamental solitons is formed
- For a clockwise rotating vortex, solitons form on the other line
- The position of the soliton set depends of the direction of the kick with respect to vortex orientation

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