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Modeling of Time-Varying (max,+) systems by means of Weighted Timed Event Graphs

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Abstract: The (max,+) theory allows to describe the behavior of Timed Event Graphs (TEG) with constant holding times. For time-varying systems, a class of First-In First-Out TEGs with periodic holding times has already been studied in the literature. We show here that such time-varying TEGs can be modeled by equivalent Weighted TEGs for which an input-output model exists. In summary, we can describe FIFO TEGs with periodic holding times by means of ultimately periodic formal series in a dioid denoted $\mathcal{E}_{per}^*[[\delta]]$.

Keywords: Weighted Timed Event Graphs, Periodic Holding Times, Dioids, Formal Power Series

1. INTRODUCTION

The class of (max,+) linear systems is of practical interest for the study and the control of traffic (railway, networks) and production systems. It is well known that they correspond to discrete event systems represented by Timed Event Graphs (TEGs) with constant holding times (see Baccelli et al. (1992), Gaubert (1992), Heidergott et al. (2006)).

Some extensions have been studied in order to increase the field of possible applications. For instance, the class of time-varying (max,+) systems has been studied in Lahaye et al. (2004) and in Lahaye et al. (2008). The authors consider Timed Event Graphs where the holding times associated to places are not constant but can have different values that change periodically. Nevertheless, tokens must not overtake: places have a First-In First-Out behavior.

Recently, the class of Weighted Timed Event Graphs (WTEGs) has been reconsidered with a dioid approach in Cottenceau et al. (2013). This work is an extension of different studies on TEG whose the arcs are valued (Cohen et al. (1998), Hamaci et al. (2004)). In (Cottenceau et al., 2013), the focus is on the modeling of WTEGs by formal power series on the basis of four elementary operators denoted respectively δ^t (time shift operator), γ^n (event shift operator), μ_m (multiplier operator) and β_b (batch operator). When considering only WTEGs such that the parallel paths have the same gain (a subclass named Weight-Balanced TEGs), it is shown in Cottenceau et al. (2013) that the transfer matrix is made of ultimately periodic series. In other words, Weight-Balanced TEGs is an extension of TEGs for which periodic phenomena is still a prevailing aspect.

We propose first to carry on the study of Weight-Balanced TEGs. In particular, we focus on the response of Single-Input Single-Output (SISO) WBTEGs to particular input trajectories called *Impulse*. We show that the impulse response, and the response to any event-shifted impulse, is an ultimately periodic counter function. Then, we show that the class of FIFO TEGs with periodic holding times studied in Lahaye et al. (2004) can be modeled by WBTEGs. For each FIFO place with a periodic holding time, we can obtain an equivalent WBTEG. Therefore, due to this model transposition, some control problems such as the ones considered in Cottenceau et al. (2001) can be solved for the FIFO TEGs as well.

2. MODELING OF WEIGHTED TIMED EVENT GRAPHS

2.1 Weighted TEGs

A Timed Event Graph (TEG) is a Petri net - with P the set of places and T the set of transition - such that each place has exactly one input and one output transition. A place can have a holding time value. In a Weighted TEG (WTEG), for each place $p_k \in P$, the edges $t_i \rightarrow p_k$ and $p_k \rightarrow t_o$ are valued by strictly positive integers denoted respectively $w_i(p_k)$ and $w_o(p_k)$. Since a WTEG is a Petri net, an initial marking denoted $M_0(p_k)$ is associated to each place $p_k \in P$. By considering only the earliest firing rule, a transition t_j is fired as soon as each input place p_l contains at least $w_o(p_l)$ token(s). Then $w_o(p_l)$ token(s) is(are) removed from each input place p_l of t_j , and $w_i(p_k)$ token(s) is(are) added to each output place p_k of t_j . In short, the weights attached to the edges describe how many tokens are produced/consumed by the transition firings.

Definition 1. (Gain of a path). Let us consider p_k a place with t_i as input transition and t_o as output transition. The elementary oriented path (through p_k) $t_i \rightarrow p_k \rightarrow t_o$

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has a gain defined by $\Gamma(p_k) = w_i(p_k)/w_o(p_k) \in \mathbb{Q}$. By extension, for a path Π passing through a set of places $p_a \rightarrow p_b \rightarrow \dots \rightarrow p_k$, the gain is defined as $\Gamma(\Pi) = \Gamma(p_a) \times \Gamma(p_b) \times \dots \times \Gamma(p_k)$.

Definition 2. (Balanced Property). A Weighted TEG is said Weight-Balanced if for each pair of transitions $(t_a, t_b) \in T \times T$, all the oriented paths from t_a to t_b have the same gain.

Example 3. For instance, the WTEG depicted on Fig. 1 is Weight-Balanced. All the paths from x_1 to x_4 have the same gain equal to $3/2$.

According to Def.2, a Weight-Balanced TEG has necessarily all its circuits (paths from t_i to t_i) with a gain equal to 1. Such circuits are said conservative. Hereafter, only Weight-Balanced TEGs (WBTEGs) are considered.

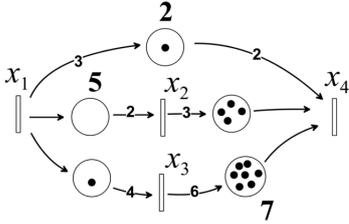


Fig. 1. Weight-Balanced TEG.

2.2 Dioid of operators

A counter function $x : t \mapsto x(t)$ describes the number of events of type x observed up to time t . By denoting Σ the set of counter functions, an operator is a map $\mathcal{H} : \Sigma \rightarrow \Sigma$ which is said *linear* if $\forall x, y \in \Sigma$, a) $\mathcal{H}(\min(x, y)) = \min(\mathcal{H}(x), \mathcal{H}(y))$ and b) $\mathcal{H}(\lambda + x) = \lambda + \mathcal{H}(x)$. An operator is said *additive* if only a) is satisfied.

Definition 4. (Dioid \mathcal{O}). The set of additive operators on Σ is denoted \mathcal{O} and is a dioid (idempotent semiring) when considering the operations defined below : $x \in \Sigma$, $\forall \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$

$$\begin{aligned} \mathcal{H}_1 \oplus \mathcal{H}_2 &\triangleq \forall x, (\mathcal{H}_1 \oplus \mathcal{H}_2)(x) = \min(\mathcal{H}_1(x), \mathcal{H}_2(x)) \\ \mathcal{H}_1 \circ \mathcal{H}_2 &\triangleq \forall x, (\mathcal{H}_1 \circ \mathcal{H}_2)(x) = \mathcal{H}_1(\mathcal{H}_2(x)). \end{aligned}$$

The null operator (neutral for \oplus and absorbing for \circ) is denoted $\varepsilon : \forall x \in \Sigma, \forall t, (\varepsilon x)(t) = +\infty$ and the unit operator (neutral for \circ) is denoted $e : \forall x \in \Sigma, \forall t, (ex)(t) = x(t)$.

Definition 5. (Order on \mathcal{O}). The canonical order relation on \mathcal{O} is: $\forall h_1, h_2 \in \mathcal{O}, h_1 \preceq h_2 \iff h_1 \oplus h_2 = h_2$.

Definition 6. (Kleene star). The Kleene star of $a \in \mathcal{O}$ is defined by $a^* = e \oplus a \oplus a^2 \dots = \bigoplus_{i \geq 0} a^i$.

The behavior of WBTEGs, with the earliest firing rule, can be modeled on dioid \mathcal{O} by combining some elementary operators given hereafter.

Definition 7. (Basic operators in WBTEGs). The next operators $\delta^t, \gamma^n, \mu_m, \beta_b \in \mathcal{O}$ are central for WBTEGs : $x \in \Sigma$,

$$\begin{aligned} \tau \in \mathbb{Z}, \delta^\tau &: \forall x, \forall t, (\delta^\tau x)(t) = x(t - \tau), \\ \nu \in \mathbb{Z}, \gamma^\nu &: \forall x, \forall t, (\gamma^\nu x)(t) = x(t) + \nu, \\ b \in \mathbb{N}^*, \beta_b &: \forall x, \forall t, (\beta_b x)(t) = \lfloor x(t)/b \rfloor, \\ m \in \mathbb{N}^*, \mu_m &: \forall x, \forall t, (\mu_m x)(t) = x(t) \times m, \end{aligned}$$

where $\lfloor a \rfloor \in \mathbb{Z}$ is the greatest integer less than or equal to $a \in \mathbb{Q}$.

Remark 8. In order to lighten formal expressions, the symbol \circ (composition) will be omitted when no confusion is possible. For instance $\gamma^3 \circ \delta^2 \circ \mu_2$ will be written simply $\gamma^3 \delta^2 \mu_2$. Let us note that the unit operator can be written $e = \gamma^0 = \delta^0 = \mu_1 = \beta_1$.

Example 9. For the WBTEG depicted on Fig. 1, we have: $x_i \in \Sigma, x_2 = \beta_2 \delta^5 x_1, x_3 = \beta_4 \gamma^1 x_1$ and $x_4 = \beta_2 \gamma^1 \delta^2 \mu_3 x_1 \oplus \gamma^4 \mu_3 x_2 \oplus \gamma^7 \mu_6 \delta^7 x_3$. With a counter description : $\forall t, x_2(t) = \lfloor x_1(t - 5)/2 \rfloor, x_3(t) = \lfloor (x_1(t) + 1)/4 \rfloor$ and $x_4(t) = \min(\lfloor (3x_1(t - 2) + 1)/2 \rfloor, 3x_2(t) + 4, 6x_3(t - 7) + 7)$.

Proposition 10. The next formal identities can be stated

$$\gamma^1 \delta^1 = \delta^1 \gamma^1; \mu_m \delta^1 = \delta^1 \mu_m; \beta_b \delta^1 = \delta^1 \beta_b \quad (1)$$

$$\mu_m \gamma^n = \gamma^{m \times n} \mu_m; \gamma^n \beta_b = \beta_b \gamma^{n \times b}. \quad (2)$$

Proof. See Cottenceau et al. (2013).

Theorem 11. For all operator $\mathcal{H} \in \mathcal{O}$, the next equalities are satisfied : $\mathcal{H} = \mathcal{H}(\delta^{-1})^* = (\delta^{-1})^* \mathcal{H} = (\gamma^1)^* \mathcal{H} = \mathcal{H}(\gamma^1)^*$.

Proof. For a counter function x , we can remark that $\forall t, x(t + 1) \geq x(t) \iff \delta^{-1} x \preceq x$. Similarly, $\forall t, x(t) + 1 \geq x(t) \iff \gamma x \preceq x$. Therefore, $\forall x \in \Sigma, \forall \mathcal{H} \in \mathcal{O}, \mathcal{H}(\gamma^1)^* x = \mathcal{H}x = (\gamma^1)^* \mathcal{H}x = \mathcal{H}(\delta^{-1})^* x = (\delta^{-1})^* \mathcal{H}x$.

The modelling of WBTEGs by means of operators leads to an input-output representation. For a given WBTEG with m inputs, p outputs, and n internal transitions, by associating a counter function to each input transition (in a vector u), to each output transition (in a vector y), and to each internal transition (in a vector x), we can describe the earliest behavior by

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases}$$

where $A \in \mathcal{O}^{n \times n}, B \in \mathcal{O}^{n \times m}$ and $C \in \mathcal{O}^{p \times n}$ are matrices whose the entries are operators. The input-output behavior of the WBTEG is then described by the rational expression

$$y = CA^* Bu = Hu. \quad (3)$$

Matrix $H \in \mathcal{O}^{p \times m}$ is called the *transfer matrix*. Entries H_{ij} are rational elements: they are finitely generated by sums, products and Kleene stars of elementary operators in $\{\gamma^n, \mu_m, \beta_b, \delta^t\}$.

2.3 Periodic Event Operators

According to (1), the delay operator δ^t can commute with any event operator, *i.e.* an operator in $\{\gamma^n, \mu_m, \beta_b\}$. Therefore, we can always write the operators involved in the behavior of WBTEGs as formal expressions $\bigoplus_i w_i \delta^{t_i}$ where w_i are operators generated by finite sums and products of event operators. In other words, the behavior of WBTEGs can be modeled by formal series in one variable δ with exponents in \mathbb{Z} and coefficients w_i on a dioid of event operators. It is why we consider separately event operators and delay operators.

Definition 12. (Dioid of E-Operators). We denote by $(\mathcal{E}, \oplus, \circ) \subset \mathcal{O}$ the subdioid of operators obtained by finite sums and products of event operators in $\{\gamma^n, \mu_m, \beta_b\}$. Elements in \mathcal{E} are called E-operators (for event operators). For instance, $w = \gamma^1 \mu_2 \oplus \beta_2 \gamma^1 \in \mathcal{E}$.

Definition 13. (Gain of $w \in \mathcal{E}$). For an E-operator, we define its gain by $\Gamma(w_a \circ w_b) = \Gamma(w_a) \times \Gamma(w_b)$ and $\Gamma(w_a \oplus$

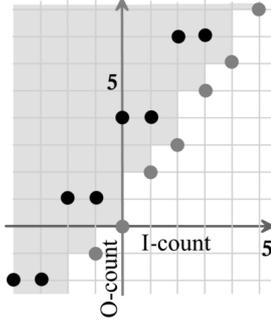


Fig. 2. Graphical representation of $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ and $\mathcal{F}_{\gamma^4\mu_3\beta_2}$

$w_b) = \min(\Gamma(w_a), \Gamma(w_b))$, with $\Gamma(\gamma^n) = 1$, $\Gamma(\mu_m) = m$ and $\Gamma(\beta_b) = 1/b$. For instance, $\Gamma(\gamma^1\mu_2\beta_3) = 2/3$.

Definition 14. (Balanced sum in \mathcal{E}). For $w_a, w_b \in \mathcal{E}$, the sum $w_a \oplus w_b$ is said balanced if $\Gamma(w_a) = \Gamma(w_b)$.

Remark 15. Due to the weight-balanced property (see Def.2), then the modeling of WBTEGs with operators involves only balanced sums.

Definition 16. (C/C function of $w \in \mathcal{E}$). We can consider an E-operator as an instantaneous system described by a Counter-to-Counter (C/C) function $\mathcal{F}_w : \mathbb{Z} \rightarrow \mathbb{Z}, k_i \mapsto k_o$, where k_i is an input counter value and k_o an output counter value. \mathcal{F}_w is obtained by replacing $x(t)$ by k_i in the expression of $(wx)(t)$.

Example 17. Let us consider $w = \beta_2\gamma^1\mu_3 \in \mathcal{E}$. Then, $(wx)(t) = \lfloor (3 \times x(t) + 1)/2 \rfloor$. The C/C function of w is $\mathcal{F}_{\beta_2\gamma^1\mu_3} : k_i \mapsto \lfloor (3 \times k_i + 1)/2 \rfloor$. This function is depicted with grey dots in Fig. 2 where the axis are labeled by I-count (input count) and O-count (output count).

Remark 18. It is important to note that $\forall w_1, w_2 \in \mathcal{E}$, then $\mathcal{F}_{w_1 \oplus w_2} = \min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2})$ and $\mathcal{F}_{w_1 \circ w_2} = \mathcal{F}_{w_1}(\mathcal{F}_{w_2})$.

Definition 19. (Periodic E-operators). An E-operator $w \in \mathcal{E}$ is said (n, n') -periodic if $\forall k \in \mathbb{Z}, \mathcal{F}_w(k+n) = \mathcal{F}_w(k) + n'$. Operator γ^n is $(1, 1)$ -periodic, μ_m is $(1, m)$ -periodic and β_b is $(b, 1)$ -periodic. The set of periodic E-operators is denoted \mathcal{E}_{per} .

Remark 20. For a (n, n') -periodic E-operator w , ratio $n'/n \in \mathbb{Q}$ is both the gain $\Gamma(w)$ and the average slope of \mathcal{F}_w .

Proposition 21. (Cottenceau et al. (2013)). For periodic E-operators, we have:

$$w_a, w_b \in \mathcal{E}_{per} \Rightarrow w_a \circ w_b \in \mathcal{E}_{per}$$

$$w_a, w_b \in \mathcal{E}_{per} \text{ and } \Gamma(w_a) = \Gamma(w_b) \Rightarrow w_a \oplus w_b \in \mathcal{E}_{per}.$$

Therefore, all the E-operators arising in the modeling of WBTEGs are periodic, since the elementary operators $\{\gamma^n, \mu_m, \beta_b\}$ are periodic and since the sums are balanced. For instance, Fig. 2 gives the representation of $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ (grey dots) and $\mathcal{F}_{\gamma^4\mu_3\beta_2}$ (black dots). Both are periodic. Moreover, we can remark that $\gamma^4\mu_3\beta_2 \preceq \beta_2\gamma^1\mu_3$ (according to Rem. 18 it is equivalent to $\mathcal{F}_{\gamma^4\mu_3\beta_2} \geq \mathcal{F}_{\beta_2\gamma^1\mu_3}$).

2.4 Transfer Series

The transfer of a SISO WBTEG can be expressed as a formal power series $\bigoplus_i w_i \delta^{t_i}$ where $w_i \in \mathcal{E}_{per}$. The considered series belongs to a dioid defined below.

Definition 22. (Dioid $\mathcal{E}_{per}[[\delta]]$). We denote by $\mathcal{E}_{per}[[\delta]]$ the dioid of formal power series in variable δ , with exponents

in \mathbb{Z} and coefficients in \mathcal{E}_{per} . The sum and the product are given by: for $s_1 = \bigoplus_i w_{1i} \delta^{t_i}$ and $s_2 = \bigoplus_j w_{2j} \delta^{t_j}$ in $\mathcal{E}_{per}[[\delta]]$, $s_1 \oplus s_2 = \bigoplus_k (w_{1k} \oplus w_{2k}) \delta^{t_k}$ and $s_1 \otimes s_2 = \bigoplus_{\tau \in \mathbb{Z}} \left(\bigoplus_{t_k + t_{k'} = \tau} w_{1k} \circ w_{2k'} \right) \delta^\tau$.

Due to property expressed in Th.11, sums and products of series in $\mathcal{E}_{per}[[\delta]]$ can be done modulo an equivalence relation. The most appropriate algebraic structure is then a quotient dioid. This quotient takes the monotonicity of counter functions into account.

Definition 23. (Quotient dioid $\mathcal{E}_{per}^*[[\delta]]$). Dioid $\mathcal{E}_{per}^*[[\delta]]$ is defined as the quotient of $\mathcal{E}_{per}[[\delta]]$ by the next equivalence

$$s_a \equiv^* s_b \iff \begin{cases} s_a(\delta^{-1})^* = s_b(\delta^{-1})^* \\ s_a(\gamma^1)^* = s_b(\gamma^1)^* \\ (\gamma^1)^* s_a = (\gamma^1)^* s_b \end{cases} \quad (4)$$

Definition 24. (Balanced series in $\mathcal{E}_{per}^*[[\delta]]$). A series $s = \bigoplus_i w_i \delta^{t_i} \in \mathcal{E}_{per}^*[[\delta]]$ is said balanced if $\forall k, \forall j, \Gamma(w_k) = \Gamma(w_j)$. For a balanced series, $\Gamma(s)$ denotes the gain of any coefficient.

Remark 25. Transfer series of WBTEGs are necessarily balanced series.

2.5 Graphical representation

A series of $\mathcal{E}_{per}^*[[\delta]]$ can be depicted graphically in a three dimensional (3D) representation. For a given series $s = \bigoplus_i w_i \delta^{t_i} \in \mathcal{E}_{per}^*[[\delta]]$, its graphical representation is depicted in a 3D basis I-count/O-count/T-shift, where each coefficient w_i is represented by its C/C function \mathcal{F}_{w_i} in the plane defined by T-shift = t_i .

Example 26. In Fig. 3, the balanced series $s = \beta_2\gamma^1\mu_3\delta^2 \oplus \gamma^4\mu_3\beta_2\delta^5 \oplus \gamma^7\mu_6\beta_4\gamma^1\delta^7$ is depicted. It is the transfer series of the WBTEG of Fig. 1. As noticed before, $\Gamma(s) = 3/2$ is the input-output gain of all the paths from x_1 to x_4 . The representation of $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ and $\mathcal{F}_{\gamma^4\mu_3\beta_2}$ is given in Fig. 2. The 3D representation is truncated to the positive values of I-count/O-count/T-shift in order to improve the readability.

In the 3D representation, products by operators δ^t and γ^n have a simple graphical interpretation.

- (Left/Right) product by $\delta^t \equiv$ a shift of t units towards the positive values of T-shift.
- Right product by $\gamma^n \equiv$ a shift of n units towards the decreasing values of I-count.
- Left product by $\gamma^{n'} \equiv$ a shift of n' units towards the increasing values of O-count.

Remark 27. The equivalence (4) can be interpreted graphically. In $\mathcal{E}_{per}^*[[\delta]]$, we have

$$w\delta^t \equiv^* (\gamma^1)^* w (\gamma^1)^* (\delta^{-1})^* \delta^t.$$

Therefore, its graphical representation is obtained by depicting \mathcal{F}_w in the plane T-shift = t and then by stretching this curve to the decreasing values of T-shift, to the decreasing values of I-count, and to the increasing values of O-count. The result is a solid volume depicted in grey.

2.6 Periodicity of WBTEGs

The transfer matrix of a WBTEG is obtained by a rational expression (3). The particularity of WBTEGs is that their

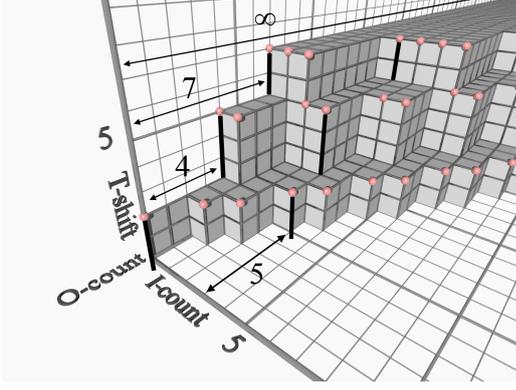


Fig. 3. Representation of $\beta_2\gamma^1\mu_3\delta^2 \oplus \gamma^4\mu_3\beta_2\delta^5 \oplus \gamma^7\mu_6\beta_4\gamma^1\delta^7$ and its response to \mathcal{I} and to $\gamma^3\mathcal{I}$.

transfer series can be put in an ultimately periodic form (see Cottencaeu et al. (2013)).

Definition 28. A balanced series $s \in \mathcal{E}_{per}^*[[\delta]]$ is said ultimately periodic if it can be written as $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ or $s = p \oplus (\gamma^{\nu'}\delta^{\tau'})^*q'$ where $\nu, \tau, \nu', \tau' \in \mathbb{N}$, p, q and q' are balanced polynomials.

Remark 29. A balanced polynomial $p = \bigoplus_{i=1}^n w_i\delta^{t_i} \in \mathcal{E}_{per}^*[[\delta]]$ can be considered as an ultimately periodic series with $\tau = \tau' = 0$.

Theorem 30. (Cottencaeu et al. (2013)). Let s_1 and s_2 be two ultimately periodic series in $\mathcal{E}_{per}^*[[\delta]]$.

- $\Gamma(s_1) = \Gamma(s_2) \Rightarrow s_1 \oplus s_2$ is ultimately periodic
- $s_1 \otimes s_2$ and $s_2 \otimes s_1$ are ultimately periodic
- $\Gamma(s_1) = 1 \Rightarrow s_1^*$ is ultimately periodic

Corollary 31. For a m -inputs p -outputs WBTEG, the entries of the transfer matrix $H = CA^*B$ are ultimately periodic series in $\mathcal{E}_{per}^*[[\delta]]$.

3. IMPULSE RESPONSE OF SINGLE-INPUT SINGLE-OUTPUT WBTEGS

3.1 Impulse Response of ordinary TEGs

In Baccelli et al. (1992), it is shown that the whole behavior of a SISO TEG can be deduced from its *impulse response*, where the impulse is the counter function $\mathcal{I} \in \Sigma$ defined as

$$\mathcal{I}(t) = \begin{cases} 0 & \text{if } t < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The output $y = H\mathcal{I}$ allows to identify the different coefficients of the transfer series H . An important feature is that the impulse response of a SISO TEG is an ultimately periodic counter :

$$\exists T, N, T_0 \text{ s.t. } t \geq T_0 \Rightarrow y(t+T) = y(t) + N.$$

As for conventional linear systems, the impulse response of a TEG gives a complete knowledge of its input-output behavior.

3.2 Impulse Response of WBTEGs

For a SISO WBTEG, it is not sufficient to observe its impulse response to have a complete knowledge of its behavior. Nevertheless, the impulse response gives a partial view of its behavior that can be obtained also from the 3D representation.

Impulse response of a monomial $w\delta^\tau$ For an operator $w\delta^\tau \in \mathcal{E}_{per}^*[[\delta]]$, the impulse response is relatively simple: $y = w\delta^\tau\mathcal{I}$ is described by the counter function $y(t) = \mathcal{I}(t - \tau) + \mathcal{F}_w(0)$, it is an impulse time-shifted by τ time units and event-shifted by $\mathcal{F}_w(0)$ units. The gain $\Gamma(w)$ is not observable on the impulse response of $w\delta^\tau$. For instance, operators $\gamma^2\delta^3$ and $\gamma^2\mu_2\delta^3$ have the same impulse response, but different gains.

Since the transfer series of a SISO WBTEG is a sum $H = \bigoplus_i w_i\delta^{t_i}$, then we can write the impulse response as

$$y = H\mathcal{I} = \bigoplus_i w_i\delta^{t_i}\mathcal{I} = \min_i \{ \mathcal{I}(t - t_i) + \mathcal{F}_{w_i}(0) \}.$$

Example 32. Let us consider the WBTEG of Fig. 1. Its transfer series is $h_1 = \beta_2\gamma^1\mu_3\delta^2 \oplus \gamma^4\mu_3\beta_2\delta^5 \oplus \gamma^7\mu_6\beta_4\gamma^1\delta^7$. Then $y_0 = h_1\mathcal{I} = h_1\gamma^0\mathcal{I}$ is a counter function given by $y_0(t < 2) = 0$, $y_0(2 \leq t < 5) = 4$, $y_0(5 \leq 7) = 7$, $y_0(t \geq 7) = \infty$. These data come from the curve depicted in the plane I-count=0 in Fig. 3. Time is along the T-shift axis and the counter value is along the O-count axis.

For the same system, if we apply an event-shifted impulse $u_3 = \gamma^3\mathcal{I}$ as input, then the output is depicted in the plane I-count=3 in Fig. 3. It is important to note that $u_3(t < 0) = 3$, $u_3(t \geq 0) = \infty$. The first 3 input events occur at $-\infty$. The response $y_3 = h_1u_3 = h_1\gamma^3\mathcal{I}$ is described by $y_3(t < 2) = 5$, $y_3(2 \leq t < 5) = 7$, $y_3(5 \leq 7) = 13$ and $y_3(t \geq 7) = \infty$.

Remark 33. In the previous example, it is important to remark that the response to $\gamma^3\mathcal{I}$ can not be obtained from the response to \mathcal{I} . Nevertheless, due to the (2,3)-periodicity of the coefficients of h_1 , then $h_1\gamma^2 = \gamma^3h_1$ and therefore $h_1\gamma^3\mathcal{I} = \gamma^3h_1\gamma^1\mathcal{I}$. The response to $\gamma^3\mathcal{I}$ is the response to $\gamma^1\mathcal{I}$ event-shifted by 3 units.

Impulse response of a WBTEG For a SISO WBTEG, the transfer series is ultimately periodic (see Th.30). It means that, in the 3D representation, the response to an event-shifted impulse $\gamma^i\mathcal{I}$ (depicted in the plane I-count= i) is also an ultimately periodic counter function.

Example 34. On Fig. 4, the 3D representation of the next series h_2 is given:

$$\begin{aligned} h_2 &= \mu_3\beta_2\delta^3 \oplus (\gamma^3\delta^4)^* (\mu_3\beta_2\gamma^1\delta^4 \oplus \gamma^3\mu_3\beta_2\delta^6) \\ &= \mu_3\beta_2\delta^3 \oplus (\mu_3\beta_2\gamma^1\delta^4) (\gamma^1\delta^2)^*. \end{aligned}$$

The ultimate left periodicity² of $h_2 \in \mathcal{E}_{per}^*[[\delta]]$ is given by $(\gamma^3\delta^4)^*$. It means that the coefficient of δ^4 is also the coefficient of δ^{4+4} , but event-shifted by 3 units (towards the increasing O-count values). The coefficient of δ^6 is also the coefficient of δ^{10} event-shifted by 3 units, and so on. The periodicity of (3,4) starts with the monomial $\mu_3\beta_2\gamma^1\delta^4$ and is described by arrows in the plane I-count=0, but the same ultimate periodicity appears in each plane I-count= i (for T-shift ≥ 4).

Moreover, in each plane T-shift= t , coefficients are (2,3)-periodic E-operators ($\Gamma(h_2) = 3/2$). Therefore, the 3D representation of h_2 is globally unchanged when it is shifted by 2 units to the increasing values of I-count and by 3 units to the increasing values of O-count. In a more

² The ultimate right periodicity (1,2) has to be searched in each plane O-count= j .

formal way, the (2, 3)-periodicity of coefficients leads to $h_2\gamma^2 = \gamma^3h_2$.

The impulse response $h_2\mathcal{I}$ is the curve depicted in the plane $I\text{-count} = 0$. The response to $\gamma^i\mathcal{I}$ (the impulse event-shifted by i unit) is depicted in the plane $I\text{-count} = i$ (a dashed line when i is an odd number). As said before, due the periodicity of E-operators, then $h_2\gamma^2\mathcal{I} = \gamma^3h_2\mathcal{I}$. The reponse to $\gamma^2\mathcal{I}$ is the response to \mathcal{I} event-shifted by 3 units. The same for $h_2\gamma^3\mathcal{I} = \gamma^3h_2\gamma^1\mathcal{I}$. Finally, it suffices to know the response to \mathcal{I} and to $\gamma^1\mathcal{I}$ to deduce $h_2\gamma^i\mathcal{I}$ when $i \geq 2$.

In summary, for an ordinary TEG, the impulse response fully characterizes its behavior (Baccelli et al. (1992)). For a SISO Weight-Balanced TEG whose the transfer series is H , all the coefficients of H being (n, n') -periodic E-operators, the impulse response is not sufficient to have a complete knowledge of H (even if the gain $\Gamma(H)$ is known). Nevertheless, the transfer H can be deduced from the reponse of a finite set of event-shifted impulses : $\{\mathcal{I}, \gamma^1\mathcal{I}, \dots, \gamma^{n-1}\mathcal{I}\}$.

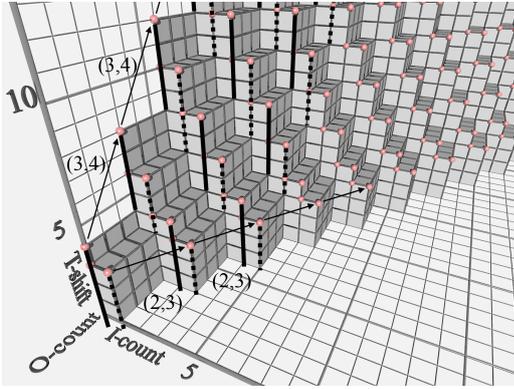


Fig. 4. Response of h_2 to $\mathcal{I}, \gamma^1\mathcal{I}, \dots, \gamma^5\mathcal{I}$

4. MODELING OF FIFO TEGS WITH TIME-VARYING HOLDING TIMES

In Lahaye et al. (2008), a FIFO place with a varying holding time is a place where a token can have a different holding time than the previous one. Nevertheless, a token can not leave a place before the previous one : a place has a First-In First-Out behavior. Here, we consider only FIFO TEGs where the holding times can change with periodical variations, as in Lahaye et al. (2004).

For a given place with u as input transition and y as output transition, we associate dater³ functions $u(k)$ and $y(k)$. We consider that the holding times $\tau(k)$ of the place depends on k and $\tau(k)$ is periodic: $\exists N \in \mathbb{N}$ s.t. $\forall k, \tau(k + N) = \tau(k)$.

Since a place has a FIFO behavior (tokens do not overtake), the output y is linked to u in the following way:

$$\begin{aligned} y(k) &= \max(\tau(k) + u(k), \tau(k-1) + u(k-1), \dots) \\ &= \max_{i \geq 0} \{\tau(k-i) + u(k-i)\}. \end{aligned}$$

Assuming that $\tau(k+N) = \tau(k)$, then $\max(\tau(k) + u(k), \tau(k-N) + u(k-N)) = \max(\tau(k) + u(k), \tau(k) + u(k-N))$.

³ $x(k)$ is the date of the $(k+1)$ -th occurrence of event x

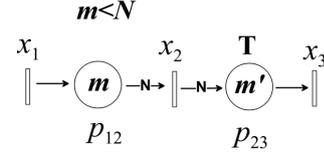


Fig. 5. WBTEG model of $x_3 = \gamma^{m'} \mu_N \beta_N \gamma^m \delta^T x_1$

$N)) = \tau(k) + u(k)$ (since $u(k+N) \geq u(k)$). Therefore, we can write

$$y(k) = \max_{0 \leq i \leq N-1} \{\tau(k-i) + u(k-i)\}. \quad (5)$$

Proposition 35. For a FIFO place with a N -periodic holding time $\tau(k)$, we can write

$$y(k) = \max_{0 \leq j \leq N-1} \left\{ \tau(j) + u \left(N \left\lfloor \frac{k-j}{N} \right\rfloor + j \right) \right\}$$

Proof. The Euclidian division of $(k-i)$ by N is

$$k-i = N \lfloor (k-i)/N \rfloor + [(k-i) \bmod N]. \quad (6)$$

Therefore, we obtain

$$\tau(k-i) = \tau((k-i) \bmod N) \quad (\text{since } \tau(k) \text{ is periodic})$$

$$u(k-i) = u(N \lfloor (k-i)/N \rfloor + [(k-i) \bmod N]).$$

According to (6), we can write $(k-i) - [(k-i) \bmod N] = N \lfloor (k-i)/N \rfloor$, which is equivalent to $\lfloor (k-i) \bmod N \rfloor = N \lfloor (k-i)/N \rfloor + i$. Therefore, we have

$$\begin{aligned} \left\lfloor \frac{k - [(k-i) \bmod N]}{N} \right\rfloor &= \left\lfloor \frac{N \lfloor (k-i)/N \rfloor + i}{N} \right\rfloor \\ &= \left\lfloor \lfloor (k-i)/N \rfloor + i/N \right\rfloor \\ &= \lfloor (k-i)/N \rfloor + \lfloor i/N \rfloor. \end{aligned}$$

When $0 \leq i \leq N-1$, then we have $\lfloor \frac{k - ((k-i) \bmod N)}{N} \rfloor = \lfloor (k-i)/N \rfloor$. By setting $j = [(k-i) \bmod N]$, then (5) can be rewritten as

$$y(k) = \max_{0 \leq j \leq N-1} \{\tau(j) + u(N \lfloor (k-j)/N \rfloor + j)\},$$

since $0 \leq i \leq N-1 \Rightarrow 0 \leq [(k-i) \bmod N] \leq N-1$.

Proposition 36. The equation $y(k) = \tau(i) + u(N \lfloor \frac{k-i}{N} \rfloor + i)$ describes the behavior of a SISO WBTEG whose the transfer series in $\mathcal{E}^*[\delta]$ is $\gamma^i \mu_N \beta_N \gamma^{N-1-i} \delta^{\tau(i)}$.

Proof. Let us consider the SISO WBTEG depicted in Fig. 5: $M_0(p_{12}) = m$ (with $m < N$), $M_0(p_{23}) = m'$ and $w_o(p_{12}) = w_i(p_{23}) = N$. Place p_{23} is timed with a constant holding time T . With the dater modeling, we obtain

$$x_2(k) = x_1(N \times k + N - 1 - m),$$

$$x_3(k) = T + x_2(\lfloor (k - m')/N \rfloor).$$

Output x_3 is linked to input x_1 by

$$x_3(k) = T + x_1(N \lfloor (k - m')/N \rfloor + N - 1 - m).$$

The same WBTEG described in $\mathcal{E}_{per}^*[\delta]$ leads to $x_3 = \gamma^{m'} \mu_N \beta_N \gamma^m \delta^T x_1$.

Corollary 37. A FIFO place with a N -periodic holding time $\tau(k)$ can be described by an equivalent WBTEG whose the transfer series is $\bigoplus_{i=0}^{N-1} \gamma^i \mu_N \beta_N \gamma^{N-1-i} \delta^{\tau(i)} \in \mathcal{E}_{per}^*[\delta]$.

Example 38. A FIFO place with a 2-periodic holding time is depicted in Fig. 6: $\tau(0) = 2$, $\tau(1) = 5$ and $\tau(k+2) = \tau(k)$. It is equivalent to a SISO WBTEG whose the transfer is $x_2 = (\mu_2 \beta_2 \gamma^1 \delta^2 \oplus \gamma^1 \mu_2 \beta_2 \delta^5) x_1$.

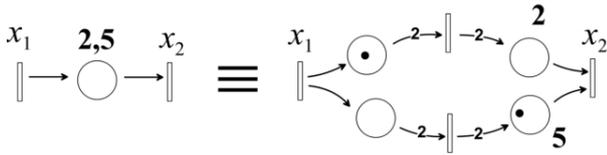


Fig. 6. A FIFO place with a periodic holding time and the equivalent WBTEG model.

Proposition 39. A SISO FIFO TEG with periodic holding times has an input-output transfer represented by a conservative ultimately periodic series in $\mathcal{E}_{per}^*[\delta]$.

Proof. Thanks to Cor. 37, a FIFO TEG with periodic holding times can be transformed into an equivalent WBTEG. Thanks to Th.30, the transfer series in $\mathcal{E}_{per}^*[\delta]$ is ultimately periodic.

5. EXAMPLE

In this section, we compute the transfer series in $\mathcal{E}_{per}^*[\delta]$ of the SISO FIFO TEG given in Fig. 7. First, we consider only the system without the output feedback loop (in dashed lines). The place $x_1 \rightarrow x_2$ has a 3-periodic holding time ($\tau_1(0) = 3, \tau_1(1) = 4, \tau_1(2) = 6$) and the place $x_2 \rightarrow y$ has a 2-periodic holding time ($\tau_2(0) = 2, \tau_2(1) = 5$). Let us

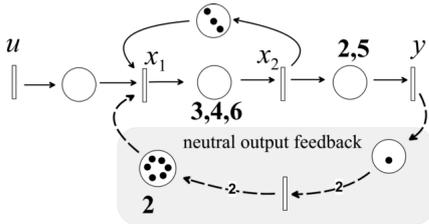


Fig. 7. A FIFO TEG

denote by $\nabla_n = \mu_n \beta_n$ a composed E-operator. The system of Fig. 7 is described by:

$$\begin{aligned} x_1 &= \gamma^0 \delta^0 u \oplus \gamma^3 x_2 \\ x_2 &= (\nabla_3 \gamma^2 \delta^3 \oplus \gamma^1 \nabla_3 \gamma^1 \delta^4 \oplus \gamma^2 \nabla_3 \delta^6) x_1 \\ y &= (\nabla_2 \gamma^1 \delta^3 \oplus \gamma^1 \nabla_2 \delta^5) x_2 \end{aligned}$$

The transfer $y = Hu$ comes from the rational expression $y = p_b p_a (\gamma^3 p_a)^* u$, with $p_a = (\nabla_3 \gamma^2 \delta^3 \oplus \gamma^1 \nabla_3 \gamma^1 \delta^4 \oplus \gamma^2 \nabla_3 \delta^6)$ and $p_b = (\nabla_2 \gamma^1 \delta^3 \oplus \gamma^1 \nabla_2 \delta^5)$. This calculus gives the next result:

$$\begin{aligned} H &= \gamma^0 \delta^5 \oplus (\gamma^1 \nabla_6 \gamma^4 \oplus \gamma^2 \nabla_6 \gamma^3 \oplus \gamma^3 \nabla_6 \gamma^2 \oplus \gamma^4 \nabla_6 \gamma^1 \oplus \gamma^5 \nabla_6) \delta^6 \\ &\oplus (\gamma^1 \nabla_6 \gamma^4 \oplus \gamma^2 \nabla_6 \gamma^3 \oplus \gamma^3 \nabla_6 \gamma^2 \oplus \gamma^4 \nabla_6) \delta^8 \\ &\oplus (\gamma^1 \nabla_6 \gamma^4 \oplus \gamma^3 \nabla_6 \gamma^3 \oplus \gamma^5 \nabla_6) \delta^9 \oplus (\gamma^3 \nabla_6 \gamma^3 \oplus \gamma^5 \nabla_6) \delta^{11} \\ &\oplus (\gamma^5 \nabla_6 \gamma^3 \oplus \gamma^7 \nabla_6 \gamma^1 \oplus \gamma^8 \nabla_6) \delta^{13} \oplus \\ &(\gamma^6 \delta^{12})^* [\gamma^5 \nabla_3 \delta^{14} \oplus (\gamma^5 \nabla_6 \gamma^3 \oplus \gamma^9 \nabla_6) \delta^{17} \oplus \\ &\gamma^8 \nabla_3 \delta^{20} \oplus (\gamma^9 \nabla_6 \gamma^3 \oplus \gamma^{11} \nabla_6) \delta^{23}] \end{aligned}$$

The ultimate periodicity is described by the term $(\gamma^6 \delta^{12})^*$ and starts with the monomial $\gamma^5 \nabla_3 \delta^{14}$. The response to any shifted impulse $\gamma^i \mathcal{I}$ is ultimately periodic: by denoting $y_i = H \gamma^i \mathcal{I}$, then $\forall t \geq 14, y_i(t+12) = y_i(t) + 6$. Finally, each coefficient of H is at least a (6,6)-periodic E-operator. Therefore, $\forall u, H \gamma^6 u = \gamma^6 H u$. Fig. 8 gives the 3D representation of H . The response to \mathcal{I} and to $\gamma^3 \mathcal{I}$ are depicted with black curves.

An input-output model is well suited to address some control problems such as the ones solved in Cottenceau

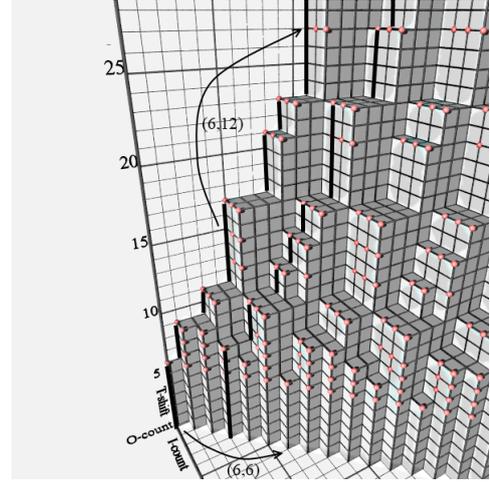


Fig. 8. Transfer series of the FIFO TEG of Fig. 7

et al. (2001) or in Maia et al. (2003). In particular, one know that for a system whose the transfer is $y = Hu$, then the greatest neutral output feedback is the system whose the transfer⁴ is $\hat{F} = H \backslash H \phi H$. System \hat{F} is the slowest feedback system that we can add between the output and the input without changing the throughput of the system. For this example, the first terms of the series $\hat{F} = H \backslash H \phi H$ are given by

$$\hat{F} = (\gamma^5 \nabla_6 \gamma^5 \oplus \gamma^8 \nabla_6 \gamma^3 \oplus \gamma^9 \nabla_6 \gamma^1) \delta^1 \oplus \gamma^6 \nabla_2 \gamma^1 \delta^2 \oplus \dots$$

For instance, $\gamma^6 \nabla_2 \gamma^1 \delta^2 \preceq \hat{F}$. Therefore, the feedback depicted on Fig. 7 (in dashed lines) is neutral for the FIFO TEG. This output feedback let the throughput unchanged (in comparison to the open loop system). The benefit from this feedback is that the path $x_1 \rightarrow y$ becomes bounded.

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⁴ \backslash and ϕ are the residuated mappings of the left and the right product (see Baccelli et al. (1992) and Cottenceau et al. (2013))

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