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QUANTISED PAINLEVÉ MONODROMY MANIFOLDS, SKLYANIN AND CALABI-YAU ALGEBRAS

LEONID CHEKHOV, MARTA MAZZOCCO, VLADIMIR RUBTSOV

to Boris Dubrovin.

Abstract. In this paper we study quantum del Pezzo surfaces belonging to a certain class. In particular we introduce the generalised Sklyanin-Painlevé algebra and characterise its PBW/PHS/Koszul properties. This algebra contains as limiting cases the generalised Sklyanin algebra, Etingof-Ginzburg and Etingof-Oblomkov-Rains quantum del Pezzo and the quantum monodromy manifolds of the Painlevé equations.

1. Introduction

In recent years, studying non-commutative rings through the methods of quantum algebraic geometry has sparked enormous interest due to its applications in mirror symmetry. The work by Gross-Hacking and Keel [21] associates to Looijenga pairs on the A-side, i.e. pairs (Y, D) where Y is a smooth projective surface and D is an anti-canonical cycle of rational curves, a mirror family on the B-side constructed as the spectrum of an explicit algebra structure on a vector space. The elements of the basis of global sections uniformise such a spectrum and are called theta functions.

Interestingly, the A-side is equipped with a symplectic structure, and it is quantised by geometric quantisation within the SYZ formalism [51], while the B side is naturally quantised by deformation quantisation.

In this paper we study a certain class of del Pezzo surfaces that can be put on either side of the mirror construction, or in other words, whose geometric and deformation quantisation coincide. In particular, we study the quantisation of a family of Poisson manifolds defined by the zero locus $\mathcal{M}_\phi$ of a degree d polynomial $\phi \in \mathbb{C}[x_1, x_2, x_3]$ of the form

$$\phi(x_1, x_2, x_3) = x_1x_2x_3 + \phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3) \tag{1.1}$$

where $\phi_i(x_i)$ for $i = 1, 2, 3$ is a polynomial of degree $\leq d$ in the variable x_i only.

From an algebro-geometric point of view (under certain conditions on the degrees of each polynomial ϕ_i, $i = 1, 2, 3$) the projective completion $\overline{\mathcal{M}}_\phi$ in the weighted projective space $\mathbb{WP}^3$ of the affine surface $\mathcal{M}_\phi \subset \mathbb{C}^3$ is a (possibly degenerate) del Pezzo surface. In other words, the pair $(\overline{\mathcal{M}}_\phi, D_\infty)$,where D_∞ is the divisor at infinity, is a Loojenga pair and $\mathcal{M}_\phi = \overline{\mathcal{M}}_\phi \setminus D_\infty$. At the same time, each affine del Pezzo surface can be considered as $\mathrm{Spec}\,(\mathbb{C}[x_1, x_2, x_3]/\langle\phi\rangle)$, which is the same as $\oplus_{k\geq 0} H^0(\mathbb{P}\mathcal{M}_\phi, L^{\otimes k})$, where $\mathbb{P}\mathcal{M}_\phi$ is the projectivisation of $\mathcal{M}_\phi$ and L is the trivial line bundle given by $\mathcal{M}_\phi \setminus \{0\}$.

A quantisation of a del Pezzo surface of this type appeared in the work of Oblomkov [32] as the spherical sub-algebra of the $\check{C}C_1$ double affine Hecke algebra (DAHA). Then Etingof, Oblomkov and Rains proposed a notion of generalised DAHA for every simply laced affine Dynkin diagram and showed that their spherical sub-algebras quantise the coordinate rings of affine surfaces obtained by removing a nodal $\mathbb{P}^1$ from a weighted projective del Pezzo surface of degrees 3, 2 and 1 respectively for $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ or by removing a triangle from a projective del Pezzo surface of degree 3 in the case $D_4^{(1)}$. In the same paper, the authors defined a holomorphic (but not algebraic) map from the mini-versal deformation of the corresponding Kleinian singularity $SL(2,\mathbb{C})/\Gamma$ (where $\Gamma \in SL(2,\mathbb{C})$ is the finite subgroup corresponding to the Dynkin diagram D_4, E_6, E_7 and E_8 respectively via the McKay correspondence) to the family of surfaces $\mathcal{M}_\phi$ where ϕ is in our form:

$$
\begin{aligned}
&D_4^{(1)} && x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \eta x_1 + \sigma x_2 + \rho x_3 + \omega,\\
&E_6^{(1)} && x_1x_2x_3 + x_1^3 + x_2^3 + x_3^2 + \eta_2 x_1^2 + \eta_1 x_1 + \sigma_2 x_2^2 + \sigma_1 x_2 + \rho x_3 + \omega,\\
&E_7^{(1)} && x_1x_2x_3 + x_1^4 + x_2^2 + x_3^2 + \eta_3 x_1^3 + \cdots + \eta_1 x_1 + \sigma x_2 + \rho x_3 + \omega,\\
&E_8^{(1)} && x_1x_2x_3 + x_1^5 + x_2^2 + x_3^2 + \eta_4 x_1^4 + \cdots + \eta_1 x_1 + \sigma x_2 + \rho x_3 + \omega.
\end{aligned}
\tag{1.2}
$$

Following this work, P. Etingof and V. Ginzburg [15] have proposed a quantum description of del Pezzo surfaces based on the flat deformation of cubic affine cone surfaces with an isolated elliptic singularity of type $\widetilde{E}_6, \widetilde{E}_7$ and $\widetilde{E}_8$ in (weighted) projective planes:

$$
\begin{aligned}
&\widetilde{E}_6 && \tau x_1x_2x_3 + \frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3} + \eta_2 x_1^2 + \eta_1 x_1 + \\
&&& \qquad + \sigma_2 x_2^2 + \sigma_1 x_2 + \rho_2 x_3^2 + \rho_1 x_3 + \omega,\\
&\widetilde{E}_7 && \tau x_1x_2x_3 + \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} + \eta_3 x_1^3 + \cdots + \eta_1 x_1 + \\
&&& \qquad + \sigma_3 x_2^3 + \cdots + \sigma_1 x_2 + \rho_2 x_3^2 + \rho_1 x_3 + \omega,\\
&\widetilde{E}_8 && \tau x_1x_2x_3 + \frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2} + \eta_5 x_1^5 + \cdots + \eta_2 x_1^2 + \eta_1 x_1 + \\
&&& \qquad + \sigma_2 x_2^2 + \sigma_1 x_2 + \rho_2 x_3^2 + \rho_1 x_3 + \omega.
\end{aligned}
\tag{1.3}
$$

Their result gives a family of Calabi-Yau algebras parametrised by a complex number and a triple of polynomials of specifically chosen degrees. Interestingly, as far as we know, nobody has proved a similar result for the polynomials (1.2).

Poisson manifolds defined by the zero locus $\mathcal{M}_\phi$ of a degree 3 polynomial $\phi \in \mathbb{C}[x_1, x_2, x_3]$ of the form (1.1) where $\phi_i(x_i)$ for $i = 1, 2, 3$ is a polynomial of degree 2 appear in the theory of the Painlevé differential equations as monodromy manifolds [54]. Indeed, the Painlevé sixth monodromy manifold is precisely the affine surface that appeared in Oblomkov [32] (see also [16]) as the spectrum of the center of the Cherednik algebra of type $\check{C}C_1$ for $q = 1$. This result was generalised in [27], where seven new algebras were produced as Whittaker degenerations of the Cherednik algebra of type $\check{C}C_1$ in such a way that their spherical–sub-algebras tend in the semi-classical limit to the monodromy manifolds of the respective Painlevé differential equations.

In the present paper we give a quantisation of the Painlevé monodromy manifolds that fits into the scheme proposed by Etingof and Ginzburg (see Theorem 1.5).

Namely, for an appropriate quantisation Φ of ϕ, we define an associative algebra A_Φ, which is a flat deformation of the coordinate ring $\mathbb{C}[x_1, x_2, x_3]$ or, more precisely, the quantisation of the corresponding Poisson algebra $A_\phi = (\mathbb{C}[x_1, x_2, x_3], \{\cdot, \cdot\}_\phi)$ where

$$\{p, q\}_\phi = \frac{dp \wedge dq \wedge d\phi}{dx_1 \wedge dx_2 \wedge dx_3}$$

is the Poisson-Nambu structure (2.11) on $\mathbb{C}^3$ for $p, q \in \mathbb{C}[x_1, x_2, x_3]$.

The algebra A_Φ has three non-commuting generators $X_i, i = 1, 2, 3$ subject to the relations

$$X_i X_j - q X_j X_i = \phi_k(X_k), \quad (i, j, k) = (1, 2, 3)$$

with $\phi_k \in \mathbb{C}[X_k]$ and $q \in \mathbb{C}^*$. One can consider the following diagram where the left and right column arrows are natural surjections and the horizontal arrows denote flat deformations or quantisations of the corresponding Poisson algebras A_ϕ and $A_\phi/(\phi)$:

$$\begin{array}{ccc} A_\phi & \overset{\text{fl. def.}}{\rightsquigarrow} & A^q_\Phi \\ \downarrow & & \downarrow \\ A_\phi/(\phi) & \overset{\text{fl. def.}}{\rightsquigarrow} & A^q_\Phi/(\Omega). \end{array} \tag{1.4}$$

Following the idea of [15], we construct the bottom-right corner algebra as a quotient of the (family of) associative algebras A^q_Φ by the bilateral ideal generated by a central element $\Omega \in A^q_\Phi$ for all ϕ corresponding to the Painlevé monodromy manifolds. As a result, we obtain a (family of) non-commutative 3-Calabi-Yau algebras that we denote by $\mathcal{UZ}$ and their non-commutative 2-dimensional quotients as a quantum del Pezzo surfaces.

More precisely we give the following:

Definition 1.1. Given any scalars $\epsilon_1, \epsilon_2, \epsilon_3$, and q, $q^m \neq 1$ for any integer m, the *universal Painlevé algebra* $\mathcal{UP}$ is the non-commutative algebra with generators $X_1, X_2, X_3, \Omega_1, \Omega_2, \Omega_3$ defined by the relations:

$$\begin{aligned} &q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1 - (q^{-1} - q)\epsilon_3 X_3 + (q^{-1/2} - q^{1/2})\Omega_3 = 0, \\ &q^{-1/2} X_2 X_3 - q^{1/2} X_3 X_2 - (q^{-1} - q)\epsilon_1 X_1 + (q^{-1/2} - q^{1/2})\Omega_1 = 0, \\ &q^{-1/2} X_3 X_1 - q^{1/2} X_1 X_3 - (q^{-1} - q)\epsilon_2 X_2 + (q^{-1/2} - q^{1/2})\Omega_2 = 0, \\ &[\Omega_i, \cdot] = 0, \qquad i = 1, 2, 3. \end{aligned} \tag{1.5}$$

Remark 1.2. The name *universal* has been chosen because in the case $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this algebra corresponds to the Universal Askey-Wilson algebra [52].

Definition 1.3. The *confluent Zhedanov algebra* $\mathcal{UZ}$ is the quotient $\mathcal{UP}/\langle \Omega_1, \Omega_2, \Omega_3 \rangle$.

Remark 1.4. The name *confluent Zhedanov* has been chosen because for different choices of the scalars $\epsilon_1, \epsilon_2, \epsilon_3$, the algebra $\mathcal{UZ}$ is coincides with the confluent Zhedanov algebras studied in [27].

Theorem 1.5. *The confluent Zhedanov algebra $\mathcal{UZ}$ satisfies the following properties:*

(1) *It is a Poincaré-Birkhoff-Witt (PBW) type deformation of the homogeneous quadratic $\mathbb{C}$-algebra with three generators X_1, X_2, X_3 and the relations:*

$$q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1 = 0,$$

$$q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 = 0,$$
$$q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 = 0. \tag{1.6}$$

(2) *It is a family of* 3*-Calabi-Yau algebras with potential*

$$\begin{aligned}\Phi_{\mathcal{UZ}} \quad &:= \quad X_1X_2X_3 - qX_2X_1X_3 + \frac{q^2-1}{2\sqrt{q}}(\epsilon_1X_1^2 + \epsilon_2X_2^2 + \epsilon_3X_3^2) + \\ &\quad +(1-q)(\Omega_1X_1 + \Omega_2X_2 + \Omega_3X_3).\end{aligned} \tag{1.7}$$

(3) *Its center* $Z(\mathcal{UZ})$ *is generated by*

(1.8)

$$\Omega_4 := \sqrt{q}X_3X_2X_1 - q\epsilon_1X_1^2 - \frac{\epsilon_2}{q}X_2^2 - q\epsilon_3X_3^2 + \sqrt{q}\Omega_1X_1 + \frac{\Omega_2}{\sqrt{q}}X_2 + \sqrt{q}\Omega_3X_3.$$

The proof of this theorem is obtained by the combining Propositions 3.5, 4.1 and 4.2. The construction of the quotient $\mathcal{UZ}/\langle\Omega_4\rangle$ within the Etingof-Ginzburg framework is carried out in Theorem 4.3.

Our quantisation is compatible with the Whittaker degeneration of generalised DAHA proposed in [27] - see Theorem 2.2 here below. In particular, we show that the Kleinian case D_4 arises as a limit of the elliptic singularity case $\widetilde{E}_6$ - all other Kleinian cases follow as special limits as well as shown in [10]. Inspired by this, we study a broad class of degenerations of Poisson algebras in terms of rational degenerations of elliptic curves.

Moreover, we connect with the work of Gross, Hacking and Keel [21], namely for each ϕ in the form (1.1) we produce a Looijenga pair (Y, D) where Y is the smooth weighted projective completion of our affine surface $\mathcal{M}_\phi \subset \mathbb{C}^3$ and D is some reduced effective normal crossing anticanonical divisor on Y given by the divisor at infinity D_∞. This is equipped with a symplectic structure obtained by taking the Poincaré residue of the global 3-form in weighted projective space $\mathbb{WP}^3$ along the divisor D_∞. This form is symplectic on $Y \setminus D_\infty = \mathcal{M}_\phi$ - this gives rise to the Nambu bracket on $\mathcal{M}_\phi$. At the same time, the coordinate ring of each affine del Pezzo $\mathcal{M}_\phi$ can be seen as the graded ring $\oplus_{k\geq 0}H^0(\mathbb{P}\mathcal{M}_\phi, L^{\otimes k})$, where $\mathbb{P}\mathcal{M}_\phi$ is the projectivisation of $\mathcal{M}_\phi$, and L is a line bundle of an appropriate degree, defined by the anticanonical divisor so that the equation $\phi = 0$ can be seen as a relation between some analogues of *theta*-functions related to toric mirror data on log-Calabi-Yau surfaces.

Due to the fact that the Calabi Yau algebra associated to $\widetilde{E}_6$ specialises to the Sklyanin algebra with three generators (4.44), we provide a unified Jacobian algebra, that we call *generalised Sklyanin-Painlevé algebra,* which for different values of the parameters specialises to the generalised Sklyanin algebra (4.45) of Iyudu and Shkarin, or to the $\widetilde{E}_6$-Calabi-Yau algebra of Etingof and Ginzburg or to our algebra $\mathcal{UZ}$.

Definition 1.6. For any choice of the scalars $a, b, c, \alpha, \beta, \gamma, a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{C}$, such that a, b, c are not roots of unity, the *generalised Sklyanin-Painlevé algebra* is the non-commutative algebra with generators X_1, X_2, X_3 defined by the relations:

$$\begin{aligned}&X_2X_3 - aX_3X_2 - \alpha X_1^2 + a_1X_1 + a_2 = 0,\\ &X_3X_1 - bX_1X_3 - \beta X_2^2 + b_1X_2 + b_2 = 0,\\ &X_1X_2 - cX_2X_1 - \gamma X_3^2 + c_1X_3 + c_2 = 0.\end{aligned} \tag{1.9}$$

We fully characterise for which cases the generalised Sklyanin-Painlevé algebra is a Calabi Yau algebra with Poincaré Birkhoff Witt (PBW) or Koszul properties or with a polynomial growth Hilbert series (PHS):

Theorem 1.7. *For specific choices of the parameters as follows:*

(1) $a = b = c \neq 0$ *and* $(a^3, \alpha\beta\gamma) \neq (-1, 1)$,
(2) $(a, b, c) \neq (0, 0, 0)$ *and either* $\alpha = \beta = a - b = 0$ *or* $\gamma = \alpha = c - a = 0$ *or* $\beta = \gamma = b - c = 0$,
(3) $\alpha = \beta = \gamma = 0$ *and* $(a, b, c) \neq (0, 0, 0)$,

the generalised Sklyanin-Painlevé algebra is potential, PHS and Koszul.

Finally, in Theorem 6.2, we deal with the question by P. Bousseau whether his deformation quantisation of function algebras on certain affine varieties related to Looijenga pairs, proposed in the recent paper [6], can be compared to Etingof and Ginzburg approach.

This paper is organised as follows. In Section 2 provide some background on the Painlevé monodromy manifolds and produce their quantisation in Theorem 1.5. In particular we introduce the family of non-commutative algebras $\mathcal{UZ}$ as the algebra generated by $\langle X_1, X_2, X_3\rangle$ and with relations (1.5). In Section 3 we discuss the notions of PBW, PHS and Koszul property and show in what way the algebra $\mathcal{UZ}$ satisfies them. In Section 4, we discuss the notions of Calabi Yau algebra, the Etingof and Ginzburg construction and the Sklyanin algebra. We introduce the generalised Sklyanin-Painlevé algebra (see subsection 4.8) and characterise its PBW/PHS/Koszul properties. In Section 5, we discuss the affine del Pezzo surfaces $\mathcal{M}_\phi$ for different choices of ϕ and their degenerations in terms of rational degenerations of elliptic curves. In Section 6 we provide the quantum version of such elliptic degenerations. Finally in Section 7 we provide several tables that resume all these results and discuss some open questions.

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Boris, your strength, energy, optimism and courage during the last months of your life are a testament to the great man you are. Goodbye dear friend, teacher.

2. Painlevé monodromy manifolds and their algebraic quantisation

The Painlevé differential equations are nonlinear second order ordinary differential equations of the type:

$$y_{tt} = \mathcal{R}(t, y, y_t),$$

where $\mathcal{R}$ is rational in y, t and y_t, such that the general solution $y(t; c_1, c_2)$ satisfies the following two important properties (see [39]):

(1) *Painlevé property:* The solutions have no movable critical points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen.
(2) *Irreducibility:* For generic values of the integration constants c_1, c_2, the solution $y(t; c_1, c_2)$ cannot be expressed via elementary or classical transcendental functions.

The Painlevé differential equations possess many beautiful properties, for example they are "integrable", i.e. they can be written as the compatibility condition

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} = [B, A], \tag{2.10}$$

between an auxiliary 2×2 linear system $\frac{\partial Y}{\partial \lambda} = A(\lambda; t)Y$ and an associated deformation system, under the condition that the monodromy data of the auxiliary system are constant under deformation.

Moreover the Painlevé differential equations admit symmetries under affine Weyl groups which are related to the associated Bäcklund transformations. Taking these into account, to each Painlevé differential equation corresponds a *monodromy manifold,* i.e. the set of monodromy data up to global conjugation and affine Weyl group symmetries. The co-called *Riemann–Hilbert correspondence* associates to each solution of a Painlevé differential equation (up to Bäcklund transformations) a point in its monodromy manifold.

Each monodromy manifold is an affine cubic surface in $\mathbb{C}^3$ defined by the zero locus of the corresponding polynomial in $\mathbb{C}[x_1, x_2, x_3]$ given in Table 1, where $\omega_1, \dots, \omega_4$ are some constants (algebraically dependent in all cases except PVI) related to the parameters appearing in the corresponding Painlevé equation.

P-eqs	Polynomials
PVI	$x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV_{deg}	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$
PIV	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
$PIII^{D_6}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$
$PIII^{D_7}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 - x_1^2 - x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2 x_2 - x_3 + \omega_4$
PII^{FN}	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

TABLE 1. Painlevé monodromy manifolds

Note that in Table 1, we distinguish ten different monodromy manifolds, the $PIII^{D_6}$, $PIII^{D_7}$ and $PIII^{D_8}$ correspond to the three different cases of the third Painlevé equation according to Sakai's classification [48], and the two monodromy manifolds PII^{FN} and PII^{JM} associated to the second Painlevé equation correspond to the two different isomonodromy problems found by Flaschka–Newell [17] and Jimbo–Miwa [24] respectively.

Each cubic surface $\mathcal{M}_\phi := \mathrm{Spec}(\mathbb{C}[x_1,x_2,x_3]/\langle \phi = 0\rangle)$, is endowed with the natural Poisson bracket defined by:

$$\{x_1,x_2\} = \frac{\partial \phi}{\partial x_3}, \quad \{x_2,x_3\} = \frac{\partial \phi}{\partial x_1}, \quad \{x_3,x_1\} = \frac{\partial \phi}{\partial x_2}. \tag{2.11}$$

In the case of PVI, this Poisson bracket is induced by the Goldman bracket on the $SL_2(\mathbb{C})$ character variety of a 4 holed Riemann sphere, or is given by the Chekhov–Fock Poisson bracket on the complexified Thurston shear coordinates. In [10], all cubic surfaces were parameterised in terms of Thurston shear coordinates s_1,s_2,s_3 and parameters p_1,p_2,p_3 such that the Poisson bracket (2.11) is induced by the following flat one:

(2.12)

$$\begin{array}{ll} \{s_1,s_2\} = \{s_2,s_3\} = \{s_3,s_1\} = 1, & \\ \{p_1,\cdot\} = \{p_2,\cdot\} = \{p_3,\cdot\} = 0, & \text{for } PVI, PV, PV_{deg}, PIV, PII, PI, \end{array}$$

$$\begin{array}{ll} \{s_1,s_2\} = \{p_2,s_1\} = \{s_3,s_2\} = \{p_2,s_3\} = 1, & \\ \{s_2,p_2\} = 2, \{s_1,s_3\} = \{p_1,\cdot\} = \{p_3,\cdot\} = 0, & \text{for } PIII^{D_6}, PIII^{D_7}, PIII^{D_8}. \end{array}$$

We give this parameterisation in Table 2, where we think of all the monodromy manifolds as having the form[1]:

$$\phi_P^{(d)} = x_1x_2x_3 - \epsilon_1^{(d)}x_1^2 - \epsilon_2^{(d)}x_2^2 - \epsilon_3^{(d)}x_3^2 + \omega_1^{(d)}x_1 + \omega_2^{(d)}x_2 + \omega_3^{(d)}x_3 + \omega_4^{(d)} = 0, \tag{2.13}$$

where d is an index running on the list of the Painlevé cubics PVI, PV, PV_{deg}, PIV, $PIII^{D_6}, PIII^{D_7}$, $PIII^{D_8}, PII^{JM}, PII^{FN}, PI$ and the parameters $\epsilon_i^{(d)}$, $\omega_i^{(d)}$, $i = 1,2,3$ are given by:

(2.14)

$$\epsilon_1^{(d)} = \begin{cases} 1 & \text{for } d = PVI, PV, PIII^{D_6}, PV_{deg}, PIII^{D_7}, PIII^{D_8}, PIV, PII^{FN}, \\ 0 & \text{for } d = PII^{JM}, PI, \end{cases}$$

$$\epsilon_2^{(d)} = \begin{cases} 1 & \text{for } d = PVI, PV, PIII^{D_6}, PV_{deg}, PIII^{D_7}, PIII^{D_8} \\ 0 & \text{for } d = PIV, PII^{FN}, PII^{JM}, PI, \end{cases}$$

$$\epsilon_3^{(d)} = \begin{cases} 1 & \text{for } d = PVI, \\ 0 & \text{for } d = PV, PIII^{D_6}, PV_{deg}, PIII^{D_7}, PIII^{D_8}, PIV, PII^{FN}, PII^{JM}, PI. \end{cases}$$

and

$$\begin{aligned} &\omega_1^{(d)} = -g_1^{(d)}g_\infty^{(d)} - \epsilon_1^{(d)}g_2^{(d)}g_3^{(d)}, \quad \omega_2^{(d)} = -g_2^{(d)}g_\infty^{(d)} - \epsilon_2^{(d)}g_1^{(d)}g_3^{(d)}, \\ &\omega_3^{(d)} = -g_3^{(d)}g_\infty^{(d)} - \epsilon_3^{(d)}g_1^{(d)}g_2^{(d)}, \\ &\omega_4^{(d)} = \epsilon_2^{(d)}\epsilon_3^{(d)}\left(g_1^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_3^{(d)}\left(g_2^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_2^{(d)}\left(g_3^{(d)}\right)^2 + \left(g_\infty^{(d)}\right)^2 + \\ &\qquad + g_1^{(d)}g_2^{(d)}g_3^{(d)}g_\infty^{(d)} - 4\epsilon_1^{(d)}\epsilon_2^{(d)}\epsilon_3^{(d)}, \end{aligned} \tag{2.15}$$

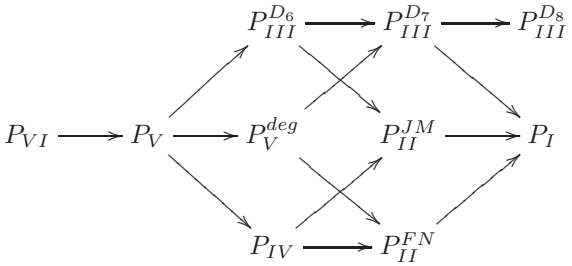
where $g_1^{(d)}, g_2^{(d)}, g_3^{(d)}, g_\infty^{(d)}$ are constants related to the parameters appearing in the Painlevé equations as described in Section 2 of [10] (note that in that paper capital letters are used for the $g_i^{(d)}$).

[1]Note that in the current paper we have inverted the signs of x_1, x_2, x_3 compared to [10].

P-eqs	Flat coordinates
PVI	$g_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad g_2 = e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}, \quad g_3 = e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}},$ $g_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}} + e^{-s_1-s_2-s_3-\frac{p_1}{2}-\frac{p_3}{2}-\frac{p_3}{2}},$ $x_1 = e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} + e^{-s_2-s_3-\frac{p_2}{2}-\frac{p_3}{2}} + e^{s_2-s_3+\frac{p_2}{2}-\frac{p_3}{2}} + g_2 e^{-s_3-\frac{p_3}{2}} + g_3 e^{s_2+\frac{p_2}{2}},$ $x_2 = e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} + e^{-s_3-s_1-\frac{p_3}{2}-\frac{p_1}{2}} + e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} + g_3 e^{-s_1-\frac{p_1}{2}} + g_1 e^{s_3+\frac{p_3}{2}},$ $x_3 = e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} + e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} + e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} + g_1 e^{-s_2-\frac{p_2}{2}} + g_2 e^{s_1+\frac{p_1}{2}}.$
PV	$g_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad g_2 = e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}, \quad g_3 = e^{-s_1-s_2-s_3-\frac{p_1}{2}-\frac{p_2}{2}}, \quad g_\infty = 1,$ $x_1 = e^{-s_1-\frac{p_1}{2}} + g_3 e^{s_2+\frac{p_2}{2}},$ $x_2 = e^{-s_2-\frac{p_2}{2}} + e^{-s_2-2s_1-\frac{p_2}{2}-p_1} + g_3 e^{-s_1-\frac{p_1}{2}} + g_1 e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}},$ $x_3 = e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} + e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} + e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} + g_1 e^{-s_2-\frac{p_2}{2}} + g_2 e^{s_1+\frac{p_1}{2}}.$
PV_{deg}	$g_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad g_2 = e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}, \quad g_3 = 0, \quad g_\infty = 1$ $x_1 = e^{-s_1-\frac{p_1}{2}}, \quad x_2 = e^{-s_2-\frac{p_2}{2}} + e^{-2s_1-s_2-p_1-\frac{p_2}{2}} + g_1 e^{-s_1-s_2-\frac{p_2}{2}-\frac{p_1}{2}},$ $x_3 = e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} + e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} + e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} + g_1 e^{-s_2-\frac{p_2}{2}} + g_2 e^{s_1+\frac{p_1}{2}}.$
PIV	$g_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad g_2 = e^{+\frac{p_2}{2}}, \quad g_3 = 0, \quad g_\infty = e^{-s_1-s_2-s_3-\frac{p_1}{2}},$ $x_1 = e^{-2s_1-s_2-2s_3-p_1} + e^{-2s_1-s_2-s_3-p_1},$ $x_2 = e^{-2s_1-s_2-p_1} + e^{-s_2} + e^{-2s_1-s_2-s_3-p_1} + g_1 e^{-s_1-s_2-\frac{p_1}{2}},$ $x_3 = e^{-s_3} + g_2 e^{s_1+\frac{p_1}{2}}.$
$PIII^{D_6}$	$g_1 = g_3 = 1, \quad g_2 = e^{s_1+\frac{s_2}{2}+\frac{p_2}{2}}, \quad g_\infty = e^{\frac{s_2}{2}+s_3+\frac{p_2}{2}},$ $x_1 = e^{-\frac{s_2}{2}+\frac{p_2}{2}} + e^{s_1-\frac{s_2}{2}+\frac{p_2}{2}} + e^{-\frac{s_2}{2}+s_3+\frac{p_2}{2}} + e^{s_1-\frac{s_2}{2}+s_3+\frac{p_2}{2}} + e^{s_1+\frac{s_2}{2}+s_3+\frac{p_2}{2}},$ $x_2 = e^{s_1} + e^{s_1-s_2} - e^{-s_2} + e^{s_3} + 2e^{s_1+s_3} + e^{-s_2+s_3} + e^{s_1-s_2+s_3} + e^{s_1+s_2+s_3},$ $x_3 = e^{-\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}+\frac{p_2}{2}}.$
$PIII^{D_7}$	$g_1 = 1, \quad g_2 = e^{s_1+\frac{s_2}{2}+\frac{p_2}{2}}, \quad x_1 = e^{-\frac{s_2}{2}+\frac{p_2}{2}} + e^{s_1-\frac{s_2}{2}+\frac{p_2}{2}} + e^{s_1+\frac{s_2}{2}+\frac{p_2}{2}},$ $g_3 = 0, \quad g_\infty = e^{s_2+\frac{p_2}{2}}, \quad x_2 = 1 + 2e^{s_1} + e^{s_1-s_2} + e^{-s_2} + e^{s_1+s_2},$ $x_3 = e^{-\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}+\frac{p_2}{2}}.$
$PIII^{D_8}$	$g_1 = 1, \quad g_2 = g_\infty = e^{\frac{s_2}{2}+\frac{p_2}{2}}, \quad g_3 = 0, \quad x_1 = e^{-\frac{s_2}{2}+\frac{p_2}{2}} + e^{\frac{s_2}{2}+\frac{p_2}{2}},$ $x_2 = 2 + e^{-s_2} + e^{s_2}, \quad x_3 = e^{-\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}-\frac{p_2}{2}} + e^{\frac{s_2}{2}+\frac{p_2}{2}}.$
PII^{JM}	$g_1 = g_3 = g_\infty = 1, \quad g_2 = e^{+\frac{p_2}{2}},$ $x_1 = e^{-s_1} + e^{-s_1-s_3}, \quad x_2 = e^{s_3} + e^{s_1+s_3}, \quad x_3 = e^{-s_2-s_3} + e^{-s_3}.$
PII^{FN}	$g_1 = e^{-s_1-s_2-s_3}, \quad g_2 = g_\infty = 1, \quad g_3 = 0, \quad x_1 = e^{s_2+s_3},$ $x_2 = e^{2s_3+s_1+s_2} + e^{2s_3+s_2} + e^{-s_1-s_2} + e^{-s_2}, \quad x_3 = e^{-s_3} + e^{-s_2-s_3}.$
PI	$g_1 = g_2 = g_\infty = 1, \quad g_3 = 0,$ $x_1 = e^{-s_1}, \quad x_2 = e^{-s_1-s_2} + e^{-s_2}, \quad x_3 = e^{s_1+s_2} + e^{s_1}.$

TABLE 2. Flat coordinates on the Painlevé monodromy manifolds

We recall that the celebrated confluence scheme of the Painlevé differential equations is the following diagram,



where the arrows represent *confluences,* i.e. degeneration procedures where the independent variable, the dependent variable and the parameters are rescaled by suitable powers of ε and then the limit $\varepsilon \to 0$ is taken. This was studied on the level of monodromy manifolds in [10].

Here, we provide the quantisation of all the Painlevé cubics and produce the corresponding *quantum confluence* in such a way that quantisation and confluence commute.

To produce the quantum Painlevé cubics, we introduce the Hermitian operators $S_1, S_2, S_3, P_1, P_2, P_3$ subject to the commutation rule inherited from the Poisson bracket of $s_1, \dots, p_3$:

$$[P_j, \cdot] = 0, \qquad [S_j, S_{j+1}] = i\pi\hbar\{s_j, s_{j+1}\} \quad j = 1, 2, 3,\ j+3 \equiv j.$$

Observe that the commutators $[S_i, S_j]$ are always numbers and therefore we have

$$\exp(aS_i)\exp(bS_j) = \exp\left(aS_i + bS_i + \frac{ab}{2}[S_i, S_j]\right),$$

for any two constants a, b. Therefore we have the Weyl ordering:

$$e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi\hbar}. \tag{2.16}$$

After quantisation, the parameters $g_1^{(d)}, \dots, g_\infty^{(d)}$ that are not equal to 0 or 1 become Hermitian operators $G_1^{(d)}, \dots, G_\infty^{(d)}$ and are automatically Casimirs. We define the operators $\Omega_i^{(d)}$ in terms of $G_1^{(d)}, \dots, G_\infty^{(d)}$ by the same formulae (2.15) that link the $\omega_i^{(d)}$ to the $g_i^{(d)}$'s - these are also Casimirs. The parameters $\epsilon_i^{(d)}$ are scalars, and they remain scalar under quantisation.

We introduce the Hermitian operators X_1, X_2, X_3 as follows: consider the classical expressions for x_1, x_2, x_3 is terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example the quantum version of $e^{s_1}e^{s_2}$ is $e^{S_1+S_2}$. Then, the following result establishes a relation between the quantisation of the Painlevé monodromy manifolds and the confluent Zhedanov algebra given in Definition 1.1:

Proposition 2.1. *The Hermitian operators $X_1, X_2, X_3, \Omega_1^{(d)}, \Omega_2^{(d)}, \Omega_3^{(d)}$ generate the algebra $\mathbb{C}\langle X_1, X_2, X_3, \Omega_1^{(d)}, \Omega_2^{(d)}, \Omega_3^{(d)}\rangle / \langle J_1, J_2, J_3, J_4\rangle$ with*

$$\begin{aligned}
J_1 &= q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 - (q^{-1}-q)\epsilon_3^{(d)}X_3 + (q^{-1/2}-q^{1/2})\Omega_3^{(d)},\\
J_2 &= q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 - (q^{-1}-q)\epsilon_1^{(d)}X_1 + (q^{-1/2}-q^{1/2})\Omega_1^{(d)},\\
J_3 &= q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 - (q^{-1}-q)\epsilon_2^{(d)}X_2 + (q^{-1/2}-q^{1/2})\Omega_2^{(d)},\\
J_4 &= [\Omega_i^{(d)}, \cdot], \qquad i = 1, 2, 3.
\end{aligned} \tag{2.17}$$

where $\epsilon_i^{(d)}$ are the same as in the classical case.

Proof. The proof of this result is obtained by direct computation by using the definitions of the quantum operators X_1, X_2 and X_3 in terms of S_1, S_2, S_3. By applying the quantum commutation relations for S_1, S_2, S_3 (2.16), relations (1.5) follow. □

In [10], we showed that the confluence procedure for the Painlevé differential equations corresponds to certain limits of the shear coordinates, for example for PVI

to PV is obtained by the substitution $p_3 \to p_3 - 2\log\varepsilon$ in the limit $\varepsilon \to 0$. We define the *quantum confluence* by the same rescaling the quantum Hermitian operators by ε and taking the same limit as $\varepsilon \to 0$. For example, by imposing exactly the same limiting procedure on P_3, we obtain a limiting procedure on the quantum operators $X_1, X_2, X_3, \Omega_1^{(VI)}, \Omega_2^{(VI)}, \Omega_3^{(VI)}$ satisfying relations (2.17) for $d = VI$, that produces some new quantum operators $X_1, X_2, X_3, \Omega_1^{(V)}, \Omega_2^{(V)}, \Omega_3^{(V)}$. By construction, these operators satisfy relations (2.17) for $d = V$. The same construction can be repeated for every d. Therefore, we have the following:

Theorem 2.2. *The confluence of the Painlevé equations commutes with their quantisation.*

3. Poincaré-Birkhoff-Witt (PBW)-deformation properties of the quantum algebra (1.5)

In this section we study the algebraic properties of the quantum algebra $\mathcal{UZ}$. The basic observation is that when all constants ϵ_i and all values of the Casimirs Ω_i, $i = 1, 2, 3$, are zero, then (1.5) are standard quantum commutation relations defining a graded algebra that is a PBW deformation of the polynomial algebra in three variables. Here we adapt the work of [41] and [7] to check that $\mathcal{UZ}$ is a PBW type deformation for all cases of ϵ_i and all values of the Casimirs Ω_i, $i = 1, 2, 3$.

3.1. **To PBW or not to PBW.** Here we discuss the definition of the PBW, PHS and Koszul properties.

Let V be a finite-dimensional $\mathbb{K}$-vector space of dimension n with basis $\{x_i\}_{i=1}^n$. Consider the tensor algebra $T^\bullet(V)$ of V over $\mathbb{K}$ - this is the free associative algebra $T^\bullet(V) = \mathbb{K}\langle x_1, \dots, x_n\rangle$. For any pair of integers $1 \le i < k \le n$ we choose an element $J_{i,k} \in T^\bullet(V)$ such that $\deg J_{i,k} \le 2$. Let J be the union of the bilateral ideals

$$x_i \otimes x_k - x_k \otimes x_i - J_{i,k}$$

in $T^\bullet(V)$. Then the quotient algebra $A = T^\bullet(V)/\langle J\rangle$ is equipped with the ascending filtration $\{F_k\}, k \ge -1; F_{-1} = 0$ (i.e. $F_{k-1} \subset F_k$) such that F_k consists of all elements of degree $\le k$ in $x_1, \dots, x_n$.

Definition 3.1. The (filtered) unital associative algebra A is said to satisfy the PBW property if there is an isomorphism of graded algebras

$$\oplus_{k\ge 0} F_k/F_{k-1} \simeq S(V),$$

where $S(V)$ is the symmetric algebra of V. [41]

Given a filtered algebra A with filtration by finite-dimensional vector spaces, we write

$$P_t(A) := \sum_{k\in\mathbb{Z}} \dim(A_k) t^k \in \mathbb{Z}[[t]]$$

for the *Hilbert-Poincaré series of the associated graded algebra*

$$\mathrm{gr}(A) = \oplus_{k\ge 0} A_k := \oplus_{k\ge 0} F_k/F_{k-1}.$$

For the purposes of this paper, we distinguish the case of $n = 3$ and give the following definition:

Definition 3.2. The algebra A is said to satisfy the PHS property if its Poincaré-Hilbert series of A coincides with $\frac{1}{(1-t)^3}$. We shall call *PHS-algebras* 3*-algebras with this property.* [22]

In the case of a Lie algebra $\mathfrak{g}$ of dimension n with a basis $\{x_1, ..., x_n\}$, there is a natural reformulation of the PBW-property for the universal enveloping algebra $U(\mathfrak{g})$ in terms of the map $\sigma : S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ where $S(\mathfrak{g})$ is the symmetric algebra of the Lie algebra $\mathfrak{g}$ and $\mathrm{gr}(U(\mathfrak{g}))$ is the associated graded algebra of the filtered algebra $T^\bullet(\mathfrak{g})$. This map is defined due to the universality of $U(\mathfrak{g})$ from the relation $\sigma \circ \tau = \phi$ where $\tau : T^\bullet(\mathfrak{g}) \to S(\mathfrak{g})$ is the canonical projection and $\phi : T^\bullet(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is the surjective morphism of graded algebras induced by the canonical projection of $T^\bullet(\mathfrak{g}) \to U(\mathfrak{g})$.

In this case, the following three statements are equivalent (see [20]):

- the homomorphism $\sigma : S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is a graded algebra isomorphism;
- if $\mathfrak{g}$ admits a totally ordered basis $\{x_\lambda\}_{\lambda\in\Lambda}$ then the subset

  $$\{1\} \cup \{x_1 \dots x_{\lambda_n} \mid (\lambda_1, \dots, \lambda_n) \in \Lambda^n, \quad \lambda_1 \leq \dots \leq \lambda_n, \quad n \geq 1\}$$

  gives a basis of $U(\mathfrak{g})$;
- the canonical map $\mathfrak{g} \to U(\mathfrak{g})$ is an injection.

We shall use these reformulations of PBW to choose among them a form which is convenient to our aims.

We conclude this subsection by recalling the definition of Koszul algebra.

Let A be a graded algebra over a field $\mathbb{K}$ of characteristic 0:

$$A = \oplus_{k=0}^{\infty} A_k,$$

its augmentation ideal A^+ is by definition

$$A^+ := \oplus_{k=1}^{\infty} A_k$$

and the canonical projection

$$\pi : A \mapsto A_0 = A/A^+,$$

is called augmentation map. By the augmentation map, A_0 can be considered as an A-module:

$$A \times A_0 \to A_0, \quad (a, x) = \pi(a)x.$$

Definition 3.3. (Koszul Algebras). A Koszul algebra A is an $\mathbb{N}$-graded algebra $A = \oplus_{k=0}^{\infty} A_k$ over a field $\mathbb{K}$ that satisfies following conditions:

- $A_0 = \mathbb{K}$.
- $A_0 \simeq A/A^+$, considered as a graded A-module, admits a graded projective resolution

  $$\dots \to P^{(2)} \to P^{(1)} \to P^{(0)} \to A_0 \to 0.$$

  such that $P^{(i)}$ is generated as a $\mathbb{Z}$-graded A-module by its degree i component, i.e., for the decomposition of A- modules:

  $$P^{(i)} = \oplus_{j\in\mathbb{Z}} P^{(i)}_j$$

  one has that $P^{(i)} = AP^{(i)}_i$.

Standard examples of Koszul algebra are the symmetric algebra $S(V)$ and the exterior algebra $\Lambda(V)$ of an n dimensional $\mathbb{K}$-vector space V.

Given a Koszul algebra $A = \oplus_{k=0}^{\infty} A_k$, consider the tensor algebra $T(A_1)$ and the map

$$\mu : T(A_1) \to A, \quad \mu(x_1 \otimes \cdots \otimes x_k) := x_1 \dots x_k.$$

A classical theorem (see for example [42]) states that every Koszul algebra is quadratic, namely,

$$A \simeq T(A_1)/\langle I \rangle$$

where $\langle I \rangle$ us the ideal generated by the quadratic relation:

$$(\ker \mu) \cap (A_1 \otimes_{\mathbb{K}} A_1).$$

The inverse statement is not always true. Priddy ([42]) proved that if a homogeneous quadratic algebra has a PBW basis, then it is Koszul.

3.2. **PBW-type algebra structure.** In this sub-section, we follow the work by Braverman-Gaitsgory [7] to adapt the ideas of sub-section 3.1 to he case of non homogeneous algebras such as our quantum algebra $\mathcal{UZ}$.

The free non-commutative polynomial associative algebra $\mathbb{C}\langle X_1, X_2, X_3\rangle$ can be considered as the tensor algebra $T^\bullet(V)$, where $V = Vect\langle X_1, X_2, X_3\rangle$, that is *filtered* by the natural filtration:

$$F^k(T^\bullet(V)) = \{\oplus_{j\leq k} T^j(V)\}.$$

We are now going to explain how this filtration descends to the quotient.

Fix a subspace $\hat{I} \subset F^2(T^\bullet(V)) = \mathbb{C} \oplus V \oplus (V \otimes V)$ and let $I \subset V \otimes V$ be the image of $\hat{I} : I = \pi(\hat{I})$ under the natural projection $\pi : F^2(T^\bullet(V)) \to V \otimes V$. There is a epimorphism of graded algebras (denoted by the same letter) $\pi : T^\bullet(V)/\langle I\rangle \to \mathrm{gr}(T^\bullet(V)/\langle \hat{I}\rangle)$.

Definition 3.4. [7] The non-homogeneous quadratic algebra $\hat{\mathcal{A}} = T^\bullet(V)/\langle \hat{I}\rangle$ *is a PBW-type deformation* of $\mathcal{A} := T^\bullet(V)/\langle I\rangle$ if the projection π is an isomorphism of graded algebras.

Roughly speaking, this means that the graded algebra $\mathrm{gr}(\hat{\mathcal{A}})$ associated to the filtered non-homogeneous quadratic algebra $\hat{\mathcal{A}} = T^\bullet(V)/\langle \hat{I}\rangle$ is the homogeneous quadratic $\mathcal{A} = T^\bullet(V)/\langle I\rangle$.

To show that our quantum algebra $\mathcal{UZ}$ is a PBW-type deformation, the first step is to show that it admits a natural filtration. This is obtained by considering it as a quotient of the free polynomial associative algebra with three generators $\mathbb{C}\langle X_1, X_2, X_3\rangle$ by non-homogeneous relations with linear and affine terms. Then, we need to prove that π is indeed an isomorphism, namely we need to prove the first statement of Theorem 1.5:

Proposition 3.5. *The quantum algebra $\mathcal{UZ}$ is a PBW type -deformation of the homogeneous quadratic $\mathbb{C}$-algebra with three generators X_1, X_2, X_3 and the relations* (1.6).

Proof. The demonstration consists of two steps. First, we drop linear and constant terms and consider the "purely" quadratic algebra $\mathcal{A}$ and show that it is a standard PBW-deformation of the polynomial free algebra $\mathbb{C}\langle X_1, X_2, X_3\rangle$ with three generators. By quotienting out the relations (1.6) we obtain a graded algebra and one can easily see (choosing, for example, the base of ordered monomials $X_1^p X_2^s X_3^r$) that

the dimension of the homogeneous components of this algebra for different $s \neq 0$ is constant (flat-deformation).

As second step, we consider the non homogeneous algebra $\hat{\mathcal{A}}$ (for generic q). This is based on the application of the following theorem due to Braverman-Gaitsgory [7] and Polishchuk-Positselsky [41] to the homogeneous ideal $\mathcal{I}(I)$ generated by the relations $I \subset V \otimes V$ (1.6):

Theorem 3.6. *Let $\hat{\mathcal{A}}$ be a non homogeneous quadratic algebra, $\hat{\mathcal{A}} = T^\bullet(V)/\langle \hat{I} \rangle$, and $\mathcal{A} = T^\bullet(V)/\langle I \rangle)$ its corresponding homogeneous quadratic algebra. Suppose $\mathcal{A}$ is a Koszul algebra. Then $\hat{\mathcal{A}}$ is a PBW-type deformation of $\mathcal{A}$ if and only if there exist linear functions $l_1 : \hat{\mathcal{A}} \to V, \quad l_2 : \hat{\mathcal{A}} \to \mathbb{C}$ for which*

$$\hat{I} = \{u - l_1(u) - l_2(u) \mid u \in I\}.$$

and the following conditions are satisfied

- $\mathrm{Im}(l_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_1) \subseteq I$;
- $l_1(l_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_1) = -(l_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_2)$,
- $l_2(l_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_1) = 0$,

where the maps $l_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_1$ and $l_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_2$ are defined on the subspace $(I \otimes V) \cap (V \otimes I) \subset T^\bullet(V)$.

Remark 3.7. For the case of the finite-dimensional Lie algebra $\mathfrak{g}$, one has $\hat{\mathcal{A}} = U(\mathfrak{g})$ and $\mathcal{A} = S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$. Consider $I \subset \mathfrak{g} \otimes \mathfrak{g}$ defined as $I = \{x_1 \otimes x_2 - x_2 \otimes x_1,\, x_1, x_2 \in \mathfrak{g}\}$. Then $l_1(x_1 \otimes x_2 - x_2 \otimes x_1) := [x_1, x_2], \quad l_2 := 0$. The three conditions in Theorem 3.6 are equivalent to the Jacobi identity.

Polishchshuk and Positselsky studied the conditions for PBW property for quadratic algebras in a more general setting ([41]). We shall reformulate the conditions in theorem 3.6 in a form that is easy to verify in our case (see Theorem 2.1 ch.5 in [41]) , i.e. in terms of the bracket operator $[\cdot,\cdot] : I \subset V \otimes V \to V$ satisfying two conditions

$$[\cdot,\cdot]_{12} - [\cdot,\cdot]_{23} : (I \otimes V) \cap (V \otimes I) \to I \tag{3.18}$$

$$[\cdot,\cdot]([\cdot,\cdot]_{12} - [\cdot,\cdot]_{23}) : (I \otimes V) \cap (V \otimes I) \mapsto 0. \tag{3.19}$$

We remark that the subspace

$$(I \otimes V) \cap (V \otimes I) \subset V \otimes V \otimes V$$

defines an analog of the space of symmetric elements of degree 3. The bracket operation $[\cdot,\cdot] : I \subset V \otimes V \to V$ is defined only on the subspace I that is why, due to the first condition ensures that the bracket maps $I \otimes V \cap V \otimes I$ again into I and we can apply it once more.

In this setting, the map $l_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_1$ is given by $[\cdot,\cdot]_{12} - [\cdot,\cdot]_{23}$ while the map $l_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes l_2$ is $[\cdot,\cdot]([\cdot,\cdot]_{12} - [\cdot,\cdot]_{23})$.

As mentioned before, if the quadratic algebra $\mathcal{A} = T^\bullet(V)/\langle I \rangle$ is Koszul then the associated graded algebra $\mathrm{gr}(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}} = T^\bullet(V)/\langle I - [\cdot,\cdot]I \rangle$ is isomorphic to $\mathrm{gr}(\mathcal{A})$. Here, $I - [\cdot,\cdot]I$ means the space of elements $u - [\cdot,\cdot]u$, $u \in I$ and the ideal $\langle I - [\cdot,\cdot]I \rangle$ coincides with the non-homogeneous ideal $\langle \hat{I} \rangle$.

Proof of Theorem 3.6. We use the conditions (3.18), (3.19) in the case when $V = \mathbb{C}X_1 \oplus \mathbb{C}X_2 \oplus \mathbb{C}X_3$ and $T^\bullet(V) = \mathbb{C}\langle X_1, X_2, X_3\rangle$ and $\langle \hat{I}\rangle$ is the ideal generated by relations (1.5):

$$\hat{\mathcal{A}} = T^\bullet(V)/\langle \hat{I}\rangle. \tag{3.20}$$

The first condition (3.18) is valid straightforwardly. The second condition (3.19) follows from the following equality

$$\begin{gathered}(X_1X_2 - qX_2X_1)X_3 + (X_2X_3 - qX_3X_2)X_1 + (X_3X_1 - qX_1X_3)X_2 = \\ X_3(X_1X_2 - qX_2X_1) + X_1(X_2X_3 - qX_3X_2) + X_2(X_3X_1 - qX_1X_3),\end{gathered} \tag{3.21}$$

that is proved by replacing the quadratic terms in the brackets by L_1, L_2, L_3, where

$$\begin{aligned} L_1 &:= (q^{-1/2} - q^{3/2})\epsilon_3^{(d)}X_3 - (1-q)\Omega_3^{(d)}, \\ L_2 &:= (q^{-1/2} - q^{3/2})\epsilon_1^{(d)}X_1 - (1-q)\Omega_1^{(d)}, \\ L_3 &:= (q^{-1/2} - q^{3/2})\epsilon_2^{(d)}X_2 - (1-q)\Omega_2^{(d)}, \end{aligned} \tag{3.22}$$

leading to the identity

$$L_1X_3 + L_2X_1 + L_3X_2 = X_3L_1 + X_1L_2 + X_2L_3,$$

that is trivially satisfied due to the fact that $[L_i, X_i] = 0$.

To conclude, the "pure quadratic" part $\mathcal{A}$ is Koszul hence, the non-homogeneous algebra $\hat{\mathcal{A}}$ is a flat deformation of the polynomial algebra $\mathbb{C}[X_1, X_2, X_3]$. □

Remark 3.8. Braverman and Gaitsgory gave a fairly simple proof that the Koszul property of $\mathcal{A}$ and the conditions (3.18) and (3.19) i.e. PBW-property imply the existence of a graded deformation $\mathcal{A}_\hbar$ of $\hat{\mathcal{A}}$ such that at $\hbar = 1$ it is *canonically* isomorphic to $\hat{\mathcal{A}}$. This is what we shall understand under "good" (or "flat") deformation properties.

3.3. **Zhedanov algebra and its degenerations.** As explained in Section 2, the quantum algebras of definition 1.3 are quantisations of the monodromy manifolds of the Painlevé differential equations. The Painlevé sixth monodromy manifold appeared in the paper by Oblomkov [32] (see also [16]) as the spectrum of the center of the Cherednik algebra of type $\check{C}_1C_1$ for $q = 1$. This result was generalised in [9] where this affine cubic surface was explicitly quantised leading to the Zhedanov algebra, which is isomorphic [25] to the spherical sub–algebra of the Cherednik algebra of type $\check{C}_1C_1$. In [27], seven new algebras were produced as confluences of the Cherednik algebra of type $\check{C}_1C_1$ in such a way that their spherical–sub-algebras tend in the semi-classical limit to the monodromy manifolds of all other Painlevé differential equations. The quantum algebras defined by relations (1.6) are isomorphic to the spherical–sub-algebras introduced in [27], in the same way in which the Zhedanov algebra is isomorphic to the spherical sub–algebra of the Cherednik algebra of type $\check{C}_1C_1$. Our Theorem 1.5 shows that the Zhedanov algebra and its degenerations are flat deformations of the polynomial algebra $\mathbb{C}[X_1, X_2, X_3]$.

4. Relation with Calabi-Yau and Sklyanin algebras

The aim of this section is to clarify the relations of our quantum algebra $\mathcal{UZ}$ with the quantum analogues of del Pezzo surfaces introduced by Etingof and Ginzburg [15]. The latter are elements of a very general class of non-commutative algebras

related to the twisted Calabi-Yau algebras introduced by V. Ginzburg [53]. We start by recalling these notions here.

4.1. **Calabi-Yau algebras and potentials.** Let A be a finite dimensional, associative and graded $\mathbb{C}$-algebra . We say that A is d-Calabi-Yau of dimension d if $\mathrm{Ext}^d_A(A, A\otimes A) \simeq A$ as a bimodule and otherwise $(n \neq d))$ $\mathrm{Ext}^n_A(A, A\otimes A) = 0$. In this paper, we will focus on the case of 3-Calabi-Yau algebras. Ginzburg has argued that most 3-Calabi-Yau algebras arise as a certain quotient of the free associative algebra. More precisely, let V be a $\mathbb{C}$-vector space with base X_1, X_2, X_3; its tensor algebra $T^\bullet(V)$ is the free associative graded algebra $A := \mathbb{C}\langle X_1, X_2, X_3\rangle$. One can consider the elements of $\mathbb{C}\langle X_1, X_2, X_3\rangle$ as non-commutative words obtained from the variables X_1, X_2, X_3. The quotient $T^\bullet(V)/[T^\bullet(V), T^\bullet(V)]$ is the space of cyclic words or "traces". This is the 0-degree Hochschild homology of the free algebra $\mathbb{C}\langle X_1, X_2, X_3\rangle$. We shall use in what follows the usual notation for the quotient of an associative algebra by the space of commutators, $A_\natural := A/[A, A]$.

One can define cyclic derivatives $\partial_j \equiv \partial_{X_j}$ for any $\Phi \in A_\natural$ by

$$\partial_j\Phi := \sum_{k|i_k=j} X_{i_k+1}X_{i_k+2}...X_{i_N}X_{i_1}X_{i_2}...X_{i_k-1} \in A, \tag{4.23}$$

where $j = 1, 2, 3$ and all indices $i_1, \ldots, i_N \in (1, 2, 3)$.

The two-sided ideal $J_\Phi =< \partial_1\Phi, \partial_2\Phi, \partial_3\Phi >$ in A is a non-commutative analogue of the Jacobian ideal and we can pass to the quotient

$$A_\Phi := A/J_\Phi. \tag{4.24}$$

We say that the an element $\Phi \in (F^3T^\bullet(V))_\natural$ is a Calabi-Yau potential if A_Φ is a 3-CY-algebra.

4.2. **Etingof-Ginzburg quantisation.** Given a polynomial $\phi \in \mathbb{C}[x_1, x_2, x_3]$, in [15]), Etingof and Ginzburg constructed an associative algebra A_Φ which is a *flat deformation* of the coordinate ring $\mathbb{C}[x_1, x_2, x_3]$ or, more precisely, the quantisation of the corresponding Poisson algebra $A_\phi = (\mathbb{C}[x_1, x_2, x_3], \{\cdot, \cdot\}_\phi)$ where

$$\{P, Q\}_\phi = \frac{dP \wedge dQ \wedge d\phi}{dx_1 \wedge dx_2 \wedge dx_3}$$

is the Poisson-Nambu structure (2.11) on $\mathbb{C}^3$ for $P, Q \in \mathbb{C}[x_1, x_2, x_3]$.

Let us remind that the flat deformations of a Poisson algebra (A, π) are governed by the second group of Poisson cohomology $HP^2(A)$ and a flatness of the Poisson algebra means also a flatness of a deformation of A as a commutative algebra. The flat deformations considered by Etingof and Ginzburg are *semiuniversal deformations* with smooth parameter scheme such that the Kodaira-Spencer map is a vector space isomorphism.

As a consequence of the computations in [38] (see Proposition 3.2), the family of affine Poisson brackets (2.11) is a family of *unimodular* Poisson brackets, so by the result of Dolgushev ([14]) the quantisation A_Φ is a Calabi-Yau algebra generated by three non-commutative generators $X_i, i = 1, 2, 3$ subject to the relations

$$\frac{\partial\Phi}{\partial X_1} = \frac{\partial\Phi}{\partial X_2} = \frac{\partial\Phi}{\partial X_3} = 0,$$

where Φ is a potential whose non commutative Jacobian ideal is a suitable quantum analogue of the classical Jacobian ideal in the local algebra of ϕ.

In [15], the authors quantise the natural Poisson structure on the hyper-surface in $\mathbb{C}^3$ with an isolated elliptic singularity of type $\tilde{E}_r$, $r = 6, 7, 8$. Such hyper-surfaces are the zero locus of the weighted homogeneous part of the polynomials (1.3) in $\mathbb{P}^2$, $\mathbb{WP}_{1,1,2}$ and $\mathbb{WP}_{1,2,3}$ respectively:

$$\begin{aligned} &\widetilde{E}_6 && \phi_\infty^{(6)} = \tau x_1x_2x_3 + \frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3},\\ &\widetilde{E}_7 && \phi_\infty^{(7)} = \tau x_1x_2x_3 + \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2},\\ &\widetilde{E}_8 && \phi_\infty^{(8)} = \tau x_1x_2x_3 + \frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2}. \end{aligned} \tag{4.25}$$

In each case, their quantisation produces a 3-Calabi-Yau algebra $A_{\Phi_\infty^{(r)}}$ defined by a suitable quantum potential $\Phi_\infty^{(r)}$, $r = 6, 7, 8$. Motivated by the study of miniversal deformations of elliptic singularities, Etingof and Ginzburg study deformations of the potential $\Phi_\infty^{(r)}$ by adding a term of the form

$$\Psi_r = P(X_1) + Q(X_2) + R(X_3), \tag{4.26}$$

where the polynomials P, Q and R depend of a total of μ arbitrary parameters, μ being the Milnor number of the elliptic singularity, and have smaller degree than $\Phi_\infty^{(r)}$ has in the variable X_1, X_2 and X_3 respectively. They prove that for such choice of Ψ_r the sum potential $\Phi_r := \Phi_\infty^{(r)} + \Psi_r$ also defines 3-Calabi-Yau algebra A_{Φ_r} with central element Ω_r that in the classical limit tends to the full polynomial ϕ_r in (1.3).

This central element, let us drop the index r to keep the discussion general, Ω is used in [15] as a non-commutative analogue of the polynomial ϕ and the quotient $A_\Phi/(\Omega)$ is a non-commutative analogue of the Poisson algebra $A_\phi/(\phi)$. As a consequence, the authors consider the following commutative diagram where the left and right column arrows are natural surjections and the wave-like arrows denote flat deformations (or quantisations) of the corresponding Poisson algebras A_ϕ and $A_\phi/(\phi)$:

$$\begin{array}{ccc} A_\phi & \overset{\text{fl. def.}}{\rightsquigarrow} & A^q_\Phi \\ \downarrow & & \downarrow \\ A_\phi/(\phi) & \overset{\text{fl. def.}}{\rightsquigarrow} & A^q_\Phi/(\Omega), \end{array} \tag{4.27}$$

The idea of [15] is to construct the bottom-right corner algebra as a quotient of the (family of) associative algebras A^q_Φ by a bilateral ideal generated by a central element $\Omega \in Z(A^q_\Phi)$.

At the quantum level, the difficulty is that the potential Φ and the central element Ω are different, even though, in their classical limit, they produce the same polynomial ϕ. As a consequence, to complete the construction of $A^q_\Phi/(\Omega)$ one needs to find the explicit expression for Ω. In [15] this was done explicitly for the elliptic singularities of type $\widetilde{E}_6, \widetilde{E}_7$ and $\widetilde{E}_8$.

Let us describe the $\tilde{E}_6$ case in some detail - as we shall see, this is the specific case that in certain limit produces the monodromy manifolds of the Painlevé equations.

It is convenient to recast the polynomial ϕ_6 in the form

$$\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d} = \tau x_1x_2x_3 + \frac{t}{3}(x_1^3+x_2^3+x_3^3) + \frac{1}{2}(a_1x_1^2+b_1x_2^2+c_1x_3^2) + a_2x_1+b_2x_2+c_2x_3+d.$$

Let us denote by $A_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}/(\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d})$ the coordinate ring defined by the affine Poisson surface $\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}=0$ in $\mathbb{C}^3$.

We note that $\phi_P^{(d)}$ defined in (2.13) is a specialisation of $\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}$ corresponding to the choice of parameters:

(4.28)

$$\tau=1,\ t=0,\ \mathbf{a}=(-2\epsilon_1^{(d)},\omega_1^{(d)}),\quad \mathbf{b}=(-2\epsilon_2^{(d)},\omega_2^{(d)}),\quad \mathbf{c}=(-2\epsilon_3^{(d)},\omega_3^{(d)}),\quad d=\omega_4^{(d)},$$

so that the $\mathbb{P}^1$ bundle over the projectivisation $\mathbb{P}M_{\phi_P}$ of the the surface M_P coincides with our general isomonodromic cubic surface (2.13).

Etingof and Ginzburg consider the family of homogeneous potentials $\Phi_{EG} \in \mathbb{C}\langle X_1,X_2,X_3\rangle_\natural$

$$\Phi_{EG} = X_1X_2X_3 - qX_2X_1X_3 - \frac{t}{3}(X_1^3+X_2^3+X_3^3), \tag{4.29}$$

and show that the filtered algebra $A^q_{\Phi_{EG}}$ with generators X_1,X_2,X_3 subject to the relations

$$\begin{aligned} X_1X_2 - qX_2X_1 &= tX_3^2,\\ X_2X_3 - qX_3X_2 &= tX_1^2,\\ X_3X_1 - qX_1X_3 &= tX_2^2, \end{aligned}$$

is a 3-CY algebra.

They then add a deformation potential Ψ_{EG} where

$$\Psi_{EG} = \frac{1}{2}(a_1X_1^2+b_1X_2^2+c_1X_3^2) + a_2X_1+b_2X_2+c_2X_3+d. \tag{4.30}$$

Note that this deformation potential is precisely in the form (4.26) with

$$P=\frac{1}{2}a_1X_1^2+a_2X_1+\frac{1}{3}d,\quad Q=\frac{1}{2}b_1X_2^2+b_2X_2+\frac{1}{3}d,\quad R=\frac{1}{2}c_1X_3^2+c_2X_3+\frac{1}{3}d.$$

The sum $\Phi_{EG}+\Psi_{EG}$ depends on q and further 8 parameters $\mathbf{a}=(a_1,a_2),\mathbf{b}=(b_1,b_2),\mathbf{c}=(c_1,c_2),q$ and t (the Milnor number of the corresponding elliptic Gorenstein singularity is 8).

The Jacobian of the potential $\Phi_{EG}+\Psi_{EG}$ gives the following relations

$$\begin{aligned} X_1X_2 - qX_2X_1 - tX_3^2 + c_1X_3 + c_2 &= 0,\\ X_2X_3 - qX_3X_2 - tX_1^2 + a_1X_1 + a_2 &= 0,\\ X_3X_1 - qX_1X_3 - tX_2^2 + b_1X_2 + b_2 &= 0, \end{aligned} \tag{4.31}$$

that define the family of algebras $A^q_{\Phi_{EG}+\Psi_{EG}}$.

The Theorem 3.4.4 from [15] claims that for generic values of the parameters $q,a_1,a_2,b_1,b_2,c_1,c_2,d,t$, the family of algebras $A^q_{\Phi_{EG}+\Psi_{EG}}$ is Calabi-Yau.

In fact, they prove a more general statement; in each case we may choose

$$\Phi_{EG} = \begin{cases} X_1X_2X_3 - qX_2X_1X_3 - \frac{t}{3}(X_1^3+X_2^3+X_3^3), & \text{for } E_6\\ X_1X_2X_3 - qX_2X_1X_3 - t\left(\frac{1}{4}X_1^4+\frac{1}{4}X_2^4+\frac{1}{2}X_3^2\right), & \text{for } E_7\\ X_1X_2X_3 - qX_2X_1X_3 - t\left(\frac{1}{6}X_1^6+\frac{1}{3}X_2^3+\frac{1}{2}X_3^2\right), & \text{for } E_8 \end{cases} \tag{4.32}$$

and taking $\Psi_{EG} = P(X_1) + Q(X_2) + R(X_3)$ depending on generic $\mu+1$ parameters with P, Q, R non-homogeneous polynomials of degree:

$$\deg(P) = \begin{cases} 2, & \text{for } E_6, \\ 3, & \text{for } E_7, \\ 5, & \text{for } E_8, \end{cases} \quad \deg(Q) = \begin{cases} 2, & \text{for } E_6, \\ 3, & \text{for } E_7, \\ 2, & \text{for } E_8, \end{cases} \quad \deg(R) = \begin{cases} 2, & \text{for } E_6, \\ 1, & \text{for } E_7, \\ 1, & \text{for } E_8, \end{cases}$$

the sum $\Phi_{EG} + \Psi_{EG}$ is a Calabi-Yau potential and its Jacobian defines a filtered family of associative 3-Calabi-Yau algebras with $\mu + 1$ parameters, where μ is the Milnor number of the respective Gorenstein singularity.

In each case $\widetilde{E}_r$, $r = 6, 7, 8$, this family of filtered algebras $A^q_{\Phi_{EG}+\Psi_{EG}}$ forms the *Rees algebras* of the corresponding algebras $A^q_{\Phi_{EG}}$ with homogeneous potentials Φ_{EG} given in (4.32). The algebras $A^q_{\Phi_{EG}+\Psi_{EG}}/(\Omega)$ where $\Omega \in Z(A^q_{\Phi_{EG}+\Psi_{EG}})$ is a non-scalar central element, give a semi-universal family of associative algebras (depending on q and μ parameters) which are 3-Calabi-Yau as well.

The theorem 3.4.5 in [15] proves that the center $Z(A^q_{\Phi_{EG}+\Psi_{EG}})$ is the polynomials algebra $\mathbb{C}[\Omega]$ and the quotient-algebra $A^q_{\Phi_{EG}+\Psi_{EG}}/\Omega$ of $A^q_{\Phi_{EG}+\Psi_{EG}}$ by the two-sided ideal $\langle\Omega\rangle$ gives a flat deformation of $A_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}/(\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d})$.

The main difficulty in this description, as it remarked by Etingof and Ginzburg, is to compute the explicit form of the Casimir Ω. In the case of $\widetilde{E}_6$, the central element is given by [15, 43]:

(4.33)

$$\begin{aligned}\Omega_{EG} = {} & (-a_1^2q^2 - a_2qt - 2a_2q^2t - a_2q^3t - b_1c_1qt^2)X_1 + \\ & t(-b_2 - 2b_2q - 2b_2q^2 - b_2q^3 - a_1c_1qt + b_1^2t^2 - b_2t^3 - b_2qt^3)X_2 + \\ & t(-c_2q - 2c_2q^2 - 2c_2q^3 - c_2q^4 - a_1b_1qt - c_1^2qt^2 + c_2t^3 + c_2qt^3)X_3 + \\ & (1+q)t^2c_1qtX_2X_1 + t(-b_1 - b_1q - b_1q^2 - 2b_1t^3 - b_1qt^3)X_2^2 + \\ & (-a_1q^2 + a_1qt^3)X_2X_3 + (1+q)t^2b_1tX_3X_1 + (a_1q^3 + a_1qt^3)X_3X_2 + \\ & t(-c_1q^2 - c_1q^3 - c_1q^4 + c_1t^3 + 2c_1qt^3)X_3^2 + (1+q)t^2(1+t)(1-t+t^2)X_2^3 + \\ & (1+q)t(q^3 - t^3)X_2X_3X_1 - (1+q)t(1+t)(1-t+t^2)qX_3X_2X_1 + (q^3 - t^3)(1+q)tX_3^3.\end{aligned}$$

4.3. **Algebra $\mathcal{UZ}$ as singular limit of an Etingof-Ginzburg Calabi-Yau algebra.** In this section we prove some further nice properties of the algebra $\mathcal{UZ}$ by showing that it is isomorphic to a singular limit of an Etingof-Ginzburg Calabi-Yau algebra. Indeed, the specialisation of relations (4.31) with

$$\begin{gathered} a_1 = \frac{(q^2-1)\epsilon_1^{(d)}}{\sqrt{q}}, \quad b_1 = \frac{(q^2-1)\epsilon_2^{(d)}}{\sqrt{q}}, \quad c_1 = \frac{(q^2-1)\epsilon_3^{(d)}}{\sqrt{q}} \\ a_2 = \Omega_1(1-q), \quad b_2 = \Omega_2(1-q), \quad c_2 = \Omega_3(1-q), \qquad t = 0, \end{gathered} \tag{4.34}$$

gives the commutation relations (1.5). The following result proves the third statement in Theorem 1.5:

Proposition 4.1. *The cubic Casimir Ω_4 defined in* (1.8) *is a special limit of the Etingof-Ginzburg central element Ω_{EG}.*

Proof. To deduce the central element Ω_4 as a limit of Ω_{EG}, we first need to introduce a quadratic term X_1^2 in Ω_{EG} by applying the commutation relations (4.31). Then,

by taking the limit as $t \to 0$ of $\frac{1}{t}(\Omega_{EG} - a_1 a_2(q^2 + t^3))$ we obtain:

(4.35)
$$\Omega_{EG}^{t=0} := (q^2-1)qX_3X_2X_1-(q+1)(a_2qX_1+b_2X_2+c_2qX_3)-a_1q^2X_1^2-b_1X_2^2-c_1q^2X_3^2.$$

The specialisation of $\Omega_{EG}^{t=0}$ with (4.34) is a central element in the algebra $\mathcal{UZ}$ that coincides with $(q^2-1)\sqrt{q}\Omega_4$. □

From this perspective, one can specialise the potential $\Phi_{EG} + \Psi_{EG}$ with the choice of parameters (4.34). In this way, one obtains precisely the potential (1.7). This potential can be decomposed as $\Phi_{\mathcal{UZ}} = \Phi_{SP} + \Psi_{\mathcal{UZ}}$ where

$$\Phi_{SP} = X_1X_2X_3 - qX_2X_1X_3 \in \mathbb{C}\langle X_1, X_2, X_3\rangle_\natural$$

is a homogeneous degree 3 potential that yields the skew polynomial algebra of three variables X_1, X_2, X_3 (1.6) and

$$\Psi_{\mathcal{UZ}} = \frac{(q^2-1)}{\sqrt{q}}\left(\epsilon_1^{(d)}X_1^2 + \epsilon_2^{(d)}X_2^2 + \epsilon_3^{(d)}X_3^2\right) + (q-1)\left(\Omega_3^{(d)}X_3 + \Omega_1^{(d)}X_1 + \Omega_2^{(d)}X_2\right),$$

is the specialisation of Ψ_{EG} with the choice of parameters (4.34). Therefore we have the following result from which the second statement of Theorem 1.5 follows automatically:

Proposition 4.2. *The associative algebra*

$$A^q_{\Phi_{\mathcal{UZ}}} := \mathbb{C}\langle X_1, X_2, X_3\rangle/\langle \partial_1\Phi_{\mathcal{UZ}}, \partial_2\Phi_{\mathcal{UZ}}, \partial_3\Phi_{\mathcal{UZ}}\rangle.$$

coincides with $\mathcal{UZ}$ and is a non-homogeneous 3-Calabi-Yau Koszul algebra.

Proof. To prove that $A^q_{\Phi_{\mathcal{UZ}}}$ coincides with $\mathcal{UZ}$ we simply observe that the cyclic derivatives of the potential $\Phi_{\mathcal{UZ}}$ give precisely the first three expressions in (1.5). To prove that $A^q_{\Phi_{\mathcal{UZ}}}$ is a 3-Calabi-Yau Koszul algebra we cannot apply Theorem 3.4.5 of [15] directly to the cubic potential (1.7) because of the fact that limit $t\to 0$ is singular. Instead, we use the fact that, as proved in Proposition 3.5, this algebra is a PBW deformation of the 3-Calabi-Yau Koszul algebra $A_{\Phi_{SP}}$ with potential Φ_{SP} and apply Theorem 3.1 in [4] that states that a non-homogeneous graded 3-algebra is a Calabi-Yau Koszul algebra if the homogeneous part is a 3-graded Calabi-Yau Koszul algebra. □

Then we can prove the following:

Theorem 4.3. *Consider the algebra The algebra $A^q_{\Phi_{\mathcal{UZ}}} \sim \mathcal{UZ}$, then the quotient $A^q_{\Phi_{\mathcal{UZ}}}/(\Omega_{\mathcal{UZ}})$ is a non-commutative deformation of the Poisson quotient $A_{\phi^{1,0}_{\epsilon,\omega}}/(\phi^{1,0}_{\epsilon,\omega})$ of the algebra $A_{\phi^{1,0}_{\epsilon,\omega}}$ and the following commutative diagram holds:*

(4.36)
$$\begin{array}{ccc} A_{\phi^{1,0}_{\epsilon,\omega}} & \overset{fl.\ def.}{\rightsquigarrow} & A^q_{\Phi_{\mathcal{UZ}}} = \mathcal{UZ} \\ \downarrow & & \downarrow \\ A_{\phi^{1,0}_{\epsilon,\omega}}/(\phi^{1,0}_{\epsilon,\omega}) & \overset{fl.\ def}{\rightsquigarrow} & A^q_{\Phi_{\mathcal{UZ}}}/(\Omega_{\mathcal{UZ}}) = \mathcal{UZ}/(\Omega_{\mathcal{UZ}}). \end{array}$$

Proof. The statements are combinations of our Theorems 1.5, 3.6 and Theorem 3.4.5 of [15]. □

4.4. **Generalised Etingof-Ginzburg cubics.** We now replace the homogeneous part Φ_{EG} given in (4.29) by $\Phi_{\alpha,\beta,\gamma} \in \mathbb{C}\langle X_1, X_2, X_3\rangle_\natural$

$$\Phi_{\alpha,\beta,\gamma} = X_1X_2X_3 - qX_2X_1X_3 - \frac{1}{3}(\alpha X_1^3 + \beta X_2^3 + \gamma X_3^3) \tag{4.37}$$

and consider the family of filtered algebras $A^q_{\Phi_{\alpha,\beta,\gamma}}$ with generators X_1, X_2, X_3 subject to the relations

$$\begin{aligned} X_1X_2 - qX_2X_1 &= \gamma X_3^2,\\ X_2X_3 - qX_3X_2 &= \alpha X_1^2,\\ X_3X_1 - qX_1X_3 &= \beta X_2^2. \end{aligned}$$

Due to the results in [22, 23] we know that algebras with homogeneous potentials from (4.37) are non-commutative Koszul 3-Calabi-Yau for certain choices of the parameters α, β, γ but they are not always PBW or PHS.

4.4.1. *Digression.* Here we list some alternative definitions of the PBW property used in the literature:[2]

Definition 4.4. The associative filtered algebra A is a PBW-algebra if

(1) The algebra A is a Koszul and has Poincaré-Hilbert series $P_A(t) = \frac{1}{(1-t)^n}$.
(2) The elements $x_1^{i_1}, x_2^{i_2}, \dots, x_n^{i_n}$, where $i_1, \dots, i_n \in \mathbb{Z}$, form a linear basis.
(3) There is an ordering on generators $x_1, \dots, x_n$ w.r.t. which the defining relations form a Gröbner basis.
(4) The associated graded algebra is canonically isomorphic to the algebra generated by the homogeneous parts of quadratic relations.

For example, the algebra of commutative polynomials satisfies (2) and in the case $n = 3$ is a PHS algebra. Note that the fourth definition implies that any homogeneous algebra automatically has the PBW property.

Example 4.5. Let A be the quantum algebra given by three generators X_1, X_2, X_3 and three relations

$$X_3^2 + aX_1X_2 + bX_2X_1, \quad X_2^2 + aX_3X_1 + bX_1X_3, \quad X_1^2 + aX_2X_3 + bX_3X_2$$

and the parameters

$$(a, b) \neq (0, 0), \quad (a^3, b^3) \neq (1, 1), \quad (a + b)^3 \neq -1.$$

This algebra (number P1, table VI in [23]) is PBW with respect to the definitions (1) (2) and (4) in 4.4 but not PBW in sense of the definition (3). Conversely, the algebra B given by three generators Y_1, Y_2, Y_3 and three relations

$$Y_1Y_2 + bY_2Y_1, \quad Y_3Y_1 + bY_1Y_3, \quad Y_2Y_3 + bY_3Y_2, \quad b \neq 0$$

(which is number PII, table VI in [23]) is a PBW-algebra for all definitions in 4.4

[2]We are in debt to Natalia Iyudu for her patient explanation and clarification of different definitions of PBW property.

4.4.2. *Calabi-Yau-Koszulity and PBW- properties of algebras whose potential is non homogeneous.* Here we consider algebras whose potential has homogeneous cubic part of $\Phi_{\alpha,\beta,\gamma}$ as well as non-homogeneous terms. Namely, we extend the family of algebras $A^q_{\Phi_{EG}+\Psi_{EG}}$ by introducing the potential $\Phi_{\alpha,\beta,\gamma}+\Psi_{EG}$ and considering the family $A^q_{\Phi_{\alpha,\beta,\gamma}+\Psi_{EG}}$ whose relations take the form

$$\begin{aligned} X_1X_2 - qX_2X_1 - \gamma X_3^2 + c_1X_3 + c_2 = 0,\\ X_2X_3 - qX_3X_2 - \alpha X_1^2 + a_1X_1 + a_2 = 0,\\ X_3X_1 - qX_1X_3 - \beta X_2^2 + b_1X_2 + b_2 = 0. \end{aligned} \tag{4.38}$$

Inspired by B. Shoikhet [49], we call this generalised algebra family by *Etingof-Ginzburg type algebras.*

The generalised Etingof-Ginzburg algebra (4.38) is a Koszul, 3-Calabi-Yau for the cases when all constants α,β,γ are equal and non-zero, or only one of them is zero, or if two of the constants are equal and non-zero but $q=1$ ([23], Table VIII).

Below, for $\gamma=0$ we have computed the central element. We stress that the corresponding Etingof-Ginzburg algebras are *not Calabi-Yau* for generic values of q, α and β.

Lemma 4.6. *For $\gamma=0$, the element*

(4.39)

$$\begin{aligned} \Omega_{GEG} = q(1+q)(-1+q^3)x_3x_2x_1 + q^3(1+q)\alpha x_1^3 + (1+q)\beta x_2^3 - a_1q^2(1+q+q^2)x_1^2\\ + c_1q(1+q)\alpha\beta x_2x_1 - b_1(1+q+q^2)x_2^2 - c_1q^2(1+q+q^2)x_3^2 -\\ - qx_1(a_2(1+2q+2q^2+q^3)+b_1c_1\alpha) + x_2(-b_2(1+2q+2q^2+q^3) - a_1c_1q\beta) -\\ qx_3(c_2(1+2q+2q^2+q^3) + c_1^2\alpha\beta) \end{aligned}$$

is a central element in the algebra (4.38).

As already mentioned, the algebra $A^q_{\Phi_{EG}+\Psi_{EG}}$ is a non-commutative Calabi-Yau algebra. Moreover, in [15] it is shown that the Hilbert-Poincaré polynomial of the algebra $A^q_{\Phi_{EG}+\Psi_{EG}}$ is $\frac{1}{(1-t)^3}$, i.e. this is a PHS-algebra. Conversely, as follows from Example 4.5, the homogeneous degree 3 part $A^q_{\Phi^d_{t,\mathbf{0},\mathbf{0},\mathbf{0}}}$ of this algebra is not a PBW-algebra in the sense of (4) in Definition 4.4.

In the next subsection we discuss a known example of the generalised Etingof–Ginzburg corresponding to $\alpha=\beta=\gamma=0$ for which the Etingof-Ginzburg type algebra is "good" Koszul Calabi-Yau.

4.5. **Odesskii algebra of Sklyanin type.** In [34], Odesskii defined a quadratic algebra $\mathfrak{O}_q$ with three generators X_1,X_2,X_3 satisfying the following relations:

$$X_1X_2 - qX_2X_1 = X_3; \quad X_2X_3 - qX_3X_2 = X_1; \quad X_3X_1 - qX_1X_3 = X_2, \tag{4.40}$$

and proved that, for generic q, the center $\mathcal{Z}(\mathfrak{O}_q)$ is generated by the following element $\Omega^q := (q^2-1)X_1X_2X_3 + X_1^2 + q^2X_2^2 + X_3^2$. When $q\to 1$ the algebra tends to the universal enveloping $U(\mathfrak{sl}_2)$. Odesskii called the algebra $\mathfrak{O}_q$ a Sklyanin type algebra.

Theorem 4.7. *The Odesskii algebra $\mathfrak{O}_q$ is a PBW deformation of the 3-Calabi-Yau Koszul algebra of skew polynomials defined by the potential*

$$\Phi_O := \Phi_{SP} - \frac{1}{2}(X_1^2+X_2^2+X_3^2) \in \mathbb{C}\langle X_1,X_2,X_3\rangle_\natural. \tag{4.41}$$

Proof. This algebra is a PBW-algebra in the sense of all definitions in 4.4 because of the good PBW-properties in all senses of its homogeneous degree 3 part (see second case in the Example 4.5). To check these properties we again apply the Theorem 3.1 in the case $N = 2$ of R. Berger et R. Taillefer [4]. □

Remark 4.8. This algebra is related to the following version of a quantised universal enveloping algebra for $\mathfrak{sl}_2$ ([29]): make a rotation in the (X_1, X_2) plane:

$$X_1 \to -X_2; \quad X_2 \to X_1; \quad X_3 \to X_3$$

and then the rescaling

$$X_1 \to (q - q^{-1})X_1; \quad X_2 \to (q - q^{-1})X_2; \quad X_3 \to (q - q^{-1})X_3, \tag{4.42}$$

maps the Odesskii algebra to the algebra with relations

$$qX_1X_2 - X_2X_1 = (q - q^{-1})X_3; \tag{4.43}$$

$$qX_2X_3 - X_3X_2 = (q - q^{-1})X_1;$$
$$qX_3X_1 - X_1X_3 = (q - q^{-1})X_2$$

and with the Casimir

$$\tilde{\Omega}_O := -qX_1X_2X_3 + q^2X_1^2 + X_2^2 + X_3^2.$$

Remark 4.9. This quantum Casimir cubic goes to the famous *Markov cubic* in the limit $q \to 1$.

4.6. **Sklyanin algebra with three generators.** One of the most famous examples of a 3-Calabi-Yau algebra is the graded associative algebra $Q_3(\mathcal{E}, a, b, c)$ which is related to a (possibly degenerate or singular) elliptic curve $\mathcal{E}$

$$Q_3(\mathcal{E}, a, b, c) = \mathbb{C}\langle X_1, X_2, X_3\rangle / J_\Phi \tag{4.44}$$

with

$$J_\Phi = \langle aX_2X_3 + bX_3X_2 + cX_1^2, \quad aX_3X_1 + bX_1X_3 + cX_2^2, \quad aX_1X_2 + bX_2X_1 + cX_3^2\rangle,$$

where $(a, b, c) \in \mathbb{C}^3$ are some parameters. This algebra is a special sub-case of the one generated by Φ_{EG} with $q = \frac{b}{a}$ and $t = \frac{c}{a}$.

Artin and Schelter [2] proved that, if the parameters $(a, b, c) \in \mathbb{C}^3$ define the homogeneous coordinates of a point in $\mathcal{E}$, this algebra satisfies the Poincare-Birkhoff-Witt condition for all definitions in 4.4 except (3) and hence it can be considered as a deformation of the polynomial ring $\mathbb{C}[x_1, x_2, x_3]$. For this reason, this algebra is often called the Artin–Schelter–Tate–Sklyanin algebra with three generators, but in this paper, for brevity, we call "Sklyanin algebra" any graded associative algebra with quadratic relations which satisfies the Poincare-Birkhoff-Witt or PHS-conditions and that can be considered as a deformation of the polynomial ring $\mathbb{C}[x_1, x_2, x_3]$. Iyudu and Shkarin [22] have proved that this algebra is a CY algebra.

4.7. **Generalised Sklyanin algebras with three generators.** Iyudu and Shkarin ([22]) introduced the *generalised Sklyanin algebra* with three generators as the following quotient of the free associative algebra

$$\tilde{Q}_3(a,b,c,\alpha,\beta,\gamma) = \mathbb{C}\langle X_1, X_2, X_3\rangle / J_{GS} \tag{4.45}$$

where

$$J_{GS} = \langle X_2X_3 - aX_3X_2 - \alpha X_1^2,\ X_3X_1 - bX_1X_3 - \beta X_2^2,\ X_1X_2 - cX_2X_1 - \gamma X_3^2\rangle,$$

where $(a,b,c,\alpha,\beta,\gamma) \in \mathbb{C}^6$ is a generic set of complex constants.

These generalised Sklyanin algebras are not always potential and in fact, for generic $(a,b,c,\alpha,\beta,\gamma)$ they have neither good PBW-properties nor Koszul properties. However, for special values of the parameters they do and a complete classification is given in the following result [22]:

Theorem 4.10. *The generalised Sklyanin algebra is PHS if and only if at least one of the following conditions is satisfied:*

1. *For $a = b = c \neq 0$ and $(a^3, \alpha\beta\gamma) \neq (-1, 1)$ - this case includes the quadratic Sklyanin algebra $Q_3(\mathcal{E}, a, c, \frac{\alpha}{3})$.*
2. *For $(a,b,c) \neq (0,0,0)$ and either $\alpha = \beta = a - b = 0$ or $\gamma = \alpha = c - a = 0$ or $\beta = \gamma = b - c = 0$.*
3. *For a specific choice of all parameters in terms of a root of unity, it is a "finite" algebra which is out of our interest.*
4. *For $a = b = c = 0$ and $\alpha\beta\gamma \neq 0$. This algebra is potential without the cubic term $X_1X_2X_3$ and is out of our interest.*
5. *For $\alpha = \beta = \gamma = 0$ and $(a,b,c) \neq (0,0,0)$, this is the case of the skew polynomial algebra.*

In all these cases, the generalised Sklyanin algebra is potential and Koszul. The potential can be written as follows:

$$\Phi_{GS} = \frac{1}{3}(\alpha X_1^3 + \beta X_2^3 + \gamma X_3^3) + \tilde{a}X_1X_2X_3 + \tilde{b}X_2X_1X_3,$$

where $\tilde{a}$ and $\tilde{b}$ depend on a, b, c, q.

4.8. **Generalised Sklyanin-Painlevé potential.** Motivated by the idea of merging together generalised Sklyanin algebra and our algebra (1.5), we consider the following generalisation of the potential of Etingof and Ginzburg to include the first two cases of Theorem 4.10:

$$\Phi = \Phi_{GS} + \Psi_{EG} \tag{4.46}$$

For the choice of parameters as in the cases of Theorem 4.10, the algebra

$$\mathcal{A}^q := C\langle X_1, X_2, X_3\rangle / J,$$

where

$$J = \langle \partial_{X_1}\Phi, \partial_{X_2}\Phi, \partial_{X_3}\Phi\rangle$$

is a *generalised Sklyanin-Painlevé algebra* (1.9) and gives a PHS- or PBW-type 3-Calabi-Yau deformation of $C\langle X_1, X_2, X_3\rangle / J_{\Phi_{GS}}$.

For special choices of the parameters $\alpha, \beta, \gamma, a, b, c, a_1, b_1, c_1, a_2, b_2, c_2$, the space of objects $\mathcal{A}^q_\natural := \mathcal{A}^q/[\mathcal{A}^q, \mathcal{A}^q]$ appears in the Physics literature to which section 6 is dedicated.

Remark 4.11. It is interesting to observe the correspondence between the conditions on the constants α, β and γ in the generalised Etingof-Ginzburg algebras and the *quasiclassical* conditions on the existence Poisson-Nambu $3d$ polynomial algebras in the the paper of L. Vinet and A. Zhedanov ([55]) where they classify various Poisson analogues of the Askey-Wilson algebras AW(3). This correspondence could be behind the fact that the potential and the central element in the generalised Etingof-Ginzburg algebras are, in general, different, despite having the same semi-classical limit.

5. Poisson structures and degenerations of elliptic curves

Motivated by the observation by M. Gross, P. Hacking and S.Keel (see Example 6.13 of [21]) that the family associated to (2.13) is a log-symplectic Calabi-Yau variety, or in other words, that the projective completion Y of (2.13) with the cubic divisor D_∞ given by a triangle of lines, is an example of a Looijenga pair, in this section we study the degenerations of a certain class of Looijenga pairs (Y, D).

In this context we need to fix some notation and assumptions to make our discussion clear. We consider the polynomials $\phi \in \mathbb{C}[x_1, x_2, x_3]$ of the form (1.2), or (1.3) or belonging to Table 1. We list all such polynomials in the first column of Table 3.

Polynomials ϕ	δ weights	ϕ_∞
$\frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2} + \tau x_1x_2x_3 + \eta_5 x_1^5 + \cdots + \omega,$	1 $(1,2,3)$	$\frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2} + \tau x_1x_2x_3$
$\frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} + \tau x_1x_2x_3 + \eta_3 x_1^3 + \cdots + \omega,$	2 $(1,1,2)$	$\frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} + \tau x_1x_2x_3$
$\frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3} + \tau x_1x_2x_3 + \eta_2 x_1^2 + \cdots + \omega,$	3 $(1,1,1)$	$\frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3} + \tau x_1x_2x_3$
$x_1x_2x_3 + x_1^5 + x_2^2 + x_3^2 + \eta_4 x_1^4 + \cdots + \omega,$	1 $(2,5,3)$	$x_1x_2x_3 + x_1^5 + x_2^2$
$x_1x_2x_3 + x_1^4 + x_2^2 + x_3^2 + \eta_3 x_1^3 + \cdots + \omega,$	2 $(1,2,1)$	$x_1x_2x_3 + x_1^4 + x_2^2$
$x_1x_2x_3 + x_1^3 + x_2^3 + x_3^2 + \eta_2 x_1^2 + \cdots + \omega,$	3 $(1,1,1)$	$x_1x_2x_3 + x_1^3 + x_2^3$
$x_1x_2x_3 + \sum_{k=1}^3 (\omega_k x_k - \epsilon_k x_k^2) + \omega_4$	3 $(1,1,1)$	$x_1x_2x_3$

Table 3. del Pezzo surfaces as Loojenga pairs - in the last row we dropped the index $^{(d)}$.

In each case, the projective completion $\overline{\mathcal{M}}_\phi$ of $\mathcal{M}_\phi$ in the weighted projective spaces $\mathbb{WP}^3$ are del Pezzo surface of degree δ [16, 15]. We denote by $(x_0, \ldots, x_3)$ the weighted homogeneous coordinates in $\mathbb{WP}^3$. We list the degree δ and the weights of the variables (x_1, x_2, x_3) in the second column - we always assume the weight of the homogeneous coordinate x_0 to be 1.

For each polynomial $\phi \in \mathbb{C}[x_1, x_2, x_3]$ in Table 3, we take the weighted homogeneous part ϕ_∞ and list it in the third column. The equation $\phi_\infty = 0$ defines a

projective curve in $\mathbb{WP}^2$. The pair $(\overline{\mathcal{M}}_\phi, D_\infty)$ is a Looijenga pair and $\overline{\mathcal{M}}_\phi \setminus D_\infty$ is the affine surface $\mathcal{M}_\phi \in \mathbb{C}^3$.

The projectivisation $\mathbb{P}\mathcal{M}_\phi$ of $\mathcal{M}_\phi$ is a projective manifold of dimension 1 embedded in $\mathbb{P}^2$ by the linear system given by sections of a line bundle of degree δ - for degenerated cubic divisors of del Pezzo degree 2 and 1 such sections are expressed via Gross-Hacking-Keel θ-functions.

The coordinate ring of $\overline{\mathcal{M}}_\phi \setminus D_\infty$ is $\mathbb{C}[x_1, x_2, x_3]/\langle\phi\rangle$, which corresponds to the cone over the projectivisation $\mathbb{P}\mathcal{M}_\phi \in \mathbb{P}^2$, namely

$$\mathbb{C}[x_1, x_2, x_3]/\langle\phi\rangle = \oplus_k H^0(\mathbb{P}\mathcal{M}_\phi, L^{\otimes k}),$$

where L is the trivial bundle of degree δ. By taking the generalisation of the Poincaré residue for weighted projective spaces (see for example, [13]) of the global 3-form in $\mathbb{WP}^3$ along the divisor D_∞, one obtains a symplectic form on the quotient $\mathbb{C}[x_1, x_2, x_3]/\langle\phi\rangle$ which descends from the Nambu bracket restricted to the symplectic leaves $\phi = 0$.

In this Section we carry out the above construction for each ϕ in Table 3. We also consider special cases and *degenerations,* namely singular limits obtained by rescaling the weighted homogeneous coordinates and taking limits of such rescaling to infinity. We show that such degenerations correspond to rational degenerations of elliptic curves.

5.1. **Degenerations of the Sklyanin algebra with three generators.** In this subsection we consider $\phi_\infty = \frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3} + \tau x_1x_2x_3$, a special case of the third row of Table 3. This case is related to the quasi classical limit of the Sklyanin algebra (4.44); namely, take a, b such that $a + b$ is proportional to $1 - q$, the quasi-classical limit $q_3(\mathcal{E}, \tau)$ (where $\tau = \frac{c}{3}$) of the Sklyanin algebra $Q_3(\mathcal{E}, a, b, c)$ carries a Poisson structure (which is also called *Poisson Sklyanin algebra).* In [35] and [40] it was shown that this Poisson algebra belongs to a family of Poisson structures on the moduli space of parabolic vector bundles of degree 3 and rank 2 on the projective space $\mathbb{P}^2$. The explicit expression for the elliptic Poisson brackets of $q_3(\mathcal{E}, \tau)$ is the natural one carried by the family of the Hesse cubics

$$\phi_\tau = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) + \tau x_1x_2x_3 = 0 \tag{5.47}$$

that define the embedding of $\mathcal{E}$ in $\mathbb{P}^2$. Namely the quadratic brackets on the affine space $\mathbb{C}^3$ which define a quadratic Poisson algebra structure on

$$A_{\phi_\tau} = \mathbb{C}[x_1, x_2, x_3]/\phi_\tau = \oplus_{k\geq 0} H^0(\phi_\tau, L^{\otimes k})$$

and L is the degree 3 line bundle over the cubic curve ϕ_τ are:

$$\{x_1, x_2\} = x_3^2 + \tau x_1x_2; \quad \{x_2, x_3\} = x_1^2 + \tau x_2x_3; \quad \{x_3, x_1\} = x_2^2 + \tau x_3x_1. \tag{5.48}$$

It a straightforward computation to check that the algebra $q_3(\mathcal{E}, \tau)$ is invariant under the Heisenberg group H_3 and unimodular (see [38]).

5.1.1. *Rational degenerations of Sklyanin Poisson algebra and triangular divisor of Painlevé projective surfaces.* A. Odesskii in [33] proposed a description of all rational degenerations for a generalisation of elliptic algebras known as Sklyanin–Odesskii–Feigin algebras, and their quasi-classical counterparts - namely rational Poisson quadratic algebras. We shall restrict ourselves to one example of it in the case of the Poisson elliptic algebra $q_3(\mathcal{E}, \tau)$. It is shown in [33] that the center of

the rational degeneration $R^1_3(-\frac{2}{3})$ of the Sklyanin algebra $Q_3(\mathcal{E}, a, b, c)$ is generated by one polynomial of degree 3 in $\mathbb{P}^2$

$$\tilde{\phi} = \frac{1}{3}y_2^3 + y_1y_2y_3.$$

Indeed, if we take the Casimir element ϕ of $q_3(\mathcal{E}, \tau)$ given by the Hesse cubic (5.47) and take the rational limit $\tau \to \infty$ which gives us the triangle configuration in Figure 1

$$\{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\}$$

in the coordinates $y_i, i = 1, 2, 3$ defined as:

$$y_1 = \sqrt{\tau}x_1, \quad y_2 = x_2, \quad y_3 = \sqrt{\tau}x_3,$$

we obtain $\tilde{\phi}$.

The same triangle configuration is the divisor at infinity of the projective completion $\bar{\phi}_P \hookrightarrow \mathbb{P}^3$ of the general Painleve cubic (2.13).

Remark 5.1. It is clear that, in the limit $\tau \to \infty$, the Poisson brackets (5.48) give the cluster Poisson structure ([18])

$$\{x_1, x_2\} = x_1x_2; \quad \{x_2, x_3\} = x_2x_3; \quad \{x_3, x_1\} = x_3x_1,$$

but in the degenerated coordinates y_1, y_2, y_3 these brackets read

$$\{y_1, y_2\} = y_1y_2; \quad \{y_2, y_3\} = y_2y_3; \quad \{y_3, y_1\} = y_2^2 + y_3y_1.$$

Because these are brackets on $\mathbb{C}^3$, they define a quadratic Poisson algebra structure on

$$A_{\tilde{\phi}} = \mathbb{C}[y_1, y_2, y_3]/\tilde{\phi} = \oplus_{k\geq 0}H^0(\tilde{\phi}, L^{\otimes k})$$

where L is the degree 3 line bundle over the cubic divisor $\frac{1}{3}y_2^3 + y_1y_2y_3 = 0$ which is the union of the line $y_2 = 0$ and the conic $\frac{1}{3}y_2^2 + y_1y_3 = 0$. The rational degeneration deforms the cluster Poisson structure. We will consider the quantum version of this in subsection 6.4.

5.1.2. **Elliptic curves in weighted projective spaces, related Sklyanin Poisson structures and their rational degenerations.** We deal first with the polynomial ϕ in the third row of Table 3. As discussed in subsection 4.2, it is convenient to write this polynomial in the form

$$\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d} = \tau x_1x_2x_3 + \frac{t}{3}(x_1^3 + x_2^3 + x_3^3) + \frac{1}{2}(a_1x_1^2 + b_1x_2^2 + c_1x_3^2) + a_2x_1 + b_2x_2 + c_2x_3 + d.$$

The projectivisation $\mathbb{P}M_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}$ of the hypersurface $M_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}$ is a curve in $\mathbb{P}^2$ and $M_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}$ can be seen as a line-bundle over $\mathbb{P}M_{\phi^{\tau,t}_{\mathbf{a},\mathbf{b},\mathbf{c},d}}$. When $t = 1, \mathbf{a} = \mathbf{b} = \mathbf{c} = d = 0$ the surface $M_{\phi^{\tau,1}_{\mathbf{0},\mathbf{0},\mathbf{0},0}}$ is an affine cone over a normally embedded elliptic curve in $\mathbb{P}^2$ of degree 3 given by the homogeneous cubic

$$\left\{\phi_\infty = \tau x_1x_2x_3 + \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) = 0\right\} \subset \mathbb{P}^2.$$

This cone surface $M_{\phi^{\tau,1}_{\mathbf{0},\mathbf{0},\mathbf{0},0}}$ is an example of a simple elliptic Gorenstein singularity ($\widetilde{E}_6$ case corresponding to the elliptic singularities list). Note that the same formula for ϕ also defines a hypersurface in $\mathbb{C}^3$ with a triple point singularity in 0.

Let us now deal with the first two lines of Table 3. Denote by $\phi_{1,1,2}$ and $\phi_{2,1,3}$ the ϕ_∞ in the second and first row respectively.

The surface

$$\left\{\phi_{1,1,2}=\tau_1 x_1x_2x_3+\frac{1}{4}x_1^4+\frac{1}{4}x_2^4+\frac{1}{2}x_3^2=0\right\}\subset\mathbb{C}^3 \tag{5.49}$$

has a double point in $\mathbb{C}^3$ that is an elliptic Gorenstein singularity of type $\widetilde{E}_7$. It defines the affine cone over a homogeneous degree 4 elliptic curve in weighted projective space $\mathbb{WP}_{1,1,2}$ defined by the same equation $\phi_{1,1,2}=0$. Similarly, the surface of type $\widetilde{E}_8$

$$\left\{\phi_{2,1,3}=\tau_2 x_1x_2x_3+\frac{1}{3}x_1^3+\frac{1}{6}x_2^6+\frac{1}{2}x_3^2=0\right\}\subset\mathbb{C}^3 \tag{5.50}$$

which is the affine cone over a homogeneous degree 6 elliptic curve in weighted projective space $\mathbb{WP}_{2,1,3}$ defined by the same equation $\phi_{2,1,3}=0$.

From an algebraic point of view the coordinate rings A_ϕ discussed in subsection 4.2, and $A_{\phi_{1,1,2}}$ and $A_{\phi_{2,1,3}}$ are graded rings such that

(1) $A_\phi=\mathbb{C}[x_1,x_2,x_3]/\phi=\oplus_{k\geq 0}H^0(\phi,L^{\otimes k})$ where L is the degree 3 line bundle over the cubic curve ϕ and the sections of L form the linear system[3] defining the embedding $\phi\hookrightarrow\mathbb{P}^2$;
(2) $A_{\phi_{1,1,2}}=\oplus_{k\geq 0}H^0(\phi_{1,1,2},L^{\otimes k})=\mathbb{C}[x_1,x_2,x_3]/\phi_{1,1,2}$, where L is the degree 2 line bundle over the nodal curve $\phi_{1,1,2}$ and the sections of L define the embedding $\phi_{1,1,2}\hookrightarrow\mathbb{WP}_{1,1,2}$.
(3) $A_{\phi_{2,1,3}}=\oplus_{k\geq 0}H^0(\phi_{2,1,3},L^{\otimes k})=\mathbb{C}[x_1,x_2,x_3]/\phi_{2,1,3}$ where L is the degree 1 line bundle over the nodal curve $\phi_{2,1,3}$ and the sections of L define the embedding $\phi_{2,1,3}\hookrightarrow\mathbb{WP}_{2,1,3}$.

We apply the same procedure of degeneration as above, namely we rescale

$$x_1\to y_1 3^{1/3},\quad x_2=-\frac{y_2}{\tau_2 2^{1/2}3^{1/3}},\quad x_3=y_3 2^{1/2},$$

and take the limit $\tau_2\to\infty$ to obtain

$$\phi_{2,1,3_0}=y_1^3+y_3^2-y_1y_2y_3.$$

The corresponding Jacobian Poisson brackets read as

$$\{y_1,y_2\}=2y_3-y_1y_2,\quad \{y_2,y_3\}=3y_1^2-y_3y_2,\quad \{y_3,y_1\}=-y_1y_3, \tag{5.51}$$

and define a Posson algebra structure on the ring

$$A_{\phi_{2,1,3_0}}:=\mathbb{C}[y_1,y_2,y_3]/\phi_{2,1,3_0}=\oplus_{k\geq 0}H^0(\phi_{2,1,3_0},L^{\otimes k}), \tag{5.52}$$

where L is degree 1 line bundle over the singular curve $\tilde{\phi}_{2,1,3_0}=0$, i.e. the rational nodal cubic of arithmetic genus 1 embedded in $\mathbb{WP}_{2,1,3}$. Then

$$M_{\phi_{2,1,3_0}}:=\mathrm{Spec}A_{\phi_{2,1,3_0}}$$

is the affine cone in $\mathbb{C}^3$ over the singular curve $\phi_{2,1,3_0}=0$.

Similarly by

$$x_1=-\frac{1}{2^{1/4}\sqrt{\tau_1}}y_1,\quad x_2=\frac{1}{2^{1/4}\sqrt{\tau_1}}y_2,\quad x_3=\sqrt{2}y_3,$$

in the limit $\tau_1\to\infty$ one has

$$\phi_{1,1,2_0}=y_3^2-y_1y_2y_3. \tag{5.53}$$

[3]An explicit construction of linear systems defined by sections of L for degree 2 and 1 in terms of appropriate theta functions similar to this case can be found, for example, in the paper [46].

The corresponding Jacobian Poisson brackets read as

$$\{y_1, y_2\} = 2y_3 - y_1y_2, \quad \{y_2, y_3\} = -y_3y_2, \quad \{y_3, y_1\} = -y_1y_3, \tag{5.54}$$

and define a Posson algebra structure on the ring

$$A_{\phi_{1,1,2_0}} := \mathbb{C}[y_1, y_2, y_3]/\phi_{1,1,2_0} = \oplus_{k\geq 0} H^0(\phi_{1,1,2}, L^{\otimes k}), \tag{5.55}$$

where L is degree 2 line bundle over $\phi_{1,1,2} = 0$, the union of two rational curves $y_3 = 0$ and $y_3 - y_1y_2 = 0$ embedded in $\mathbb{WP}_{1,1,2}$ and

$$M_{\phi_{1,1,2_0}} = \mathrm{Spec} A_{\phi_{1,1,2_0}}$$

is the affine cone in $\mathbb{C}^3$.

In section 6.3 we provide a quantisation of these two degenerate cases and calculate the central elements - the quantisation of the full non-degenerate case can be found in [15].

Note that in the weighted projective space, there are many different homogeneous polynomials ϕ of degree 4 that define the same quotient by the Jacobian ideal. For example

$$\big\{\tilde\phi_{1,1,2} = \tau_1\tilde x_1\tilde x_2\tilde x_3 + \frac{1}{3}(\tilde x_3^2 + \tilde x_3\tilde x_1^2 + \tilde x_1\tilde x_2^3) = 0\big\} \subset \mathbb{WP}_{1,1,2}, \tag{5.56}$$

defines the same algebra as $\phi_{1,1,2}$ and

$$\big\{\tilde\phi_{2,1,3} = \tau_2\tilde x_1\tilde x_2\tilde x_3 + \frac{1}{3}(\tilde x_1^3 + \tilde x_2^3\tilde x_3 + \tilde x_3^2) = 0\big\} \subset \mathbb{WP}_{2,1,3}, \tag{5.57}$$

defines the same algebra as $\phi_{2,1,3}$.

In [36], A. Odesskii and the third author described two *non-rational* Poisson morphisms between the Poisson algebra of Jacobian type associated with the Hesse cubic (5.48) in the variables and the two homogeneous polynomials $\phi_{1,1,2}$:

$$\tilde x_1 = x_1^{\frac{3}{2}} x_3^{-\frac{3}{4}}, \quad \tilde x_2 = x_2 x_3^{\frac{1}{4}} x_1^{-\frac{1}{2}}, \quad y_3 = x_3^{\frac{1}{2}} \tag{5.58}$$

and $\phi_{2,1,3}$:

$$\tilde x_1 = x_1, \quad \tilde x_2 = x_2 x_3^{-\frac{1}{2}}, \quad \tilde x_3 = x_3^{\frac{3}{2}}. \tag{5.59}$$

As discussed in [36], the non-rational Poisson morphisms (5.58), (5.59) have their origin in the Calabi-Yau mirror symmetry dualities ([19]) and the question of their "quantum" interpretation was posed. Because by rescaling $\tilde x_1, \tilde x_2, \tilde x_3$ in exactly the same way as x_1, x_2, x_3 one can produce the same rational limits $\phi_{2,1,3_0}$, $\phi_{1,1,2_0}$, the quantisation produced in subsection 6.3 gives a partial answer to this question by providing a quantisation for some rational limits of $\tilde\phi_{1,1,2}$ and $\tilde\phi_{2,1,3}$.

5.1.3. **Degnerate Sklyanin algebras with three generators.** The Sklyanin algebra $Q_3(\mathcal{E}, a, b, c)$ has the following *degeneration locus*

$$\mathcal{D} = \{(1,0,0); (0,0,1); (0,0,1)\} \sqcup \{(a,b,c) \mid a^3 = b^3 = c^3\}.$$

Following [50], we call *degenerate Sklyanin algebra* the algebra $Q_3(\mathcal{E}, a, b, c)$ with $(a, b, c) \in D$.

It was proven by P. Smith that such a degenerate Sklyanin algebra is isomorphic to $\mathbb{C}\langle u, v, w\rangle/J$ where the ideal J is $J = \langle u^2 = v^2 = w^2 = 0\rangle$ if $a = b$, and $J = \langle uv = vw = wu = 0\rangle$ if $a \neq b$. In the semiclassical limit the latter case corresponds to $\phi = uvw$, which is the decorated character variety of $\pi_1(\mathbb{P}^1 \setminus \{z_1, z_2, z_3\})$ [10].

Remark 5.2. The latter model has a quiver representation with potential $Q = uu^* + vv^* + ww^* - uvw - wvu$ [8].

6. Non-commutative cubics and QFT deformations

There is an interesting similarity between the formulae for the quantum potential Φ defined in (4.46) and the non-commutative potentials describing the *marginal and relevant* deformations of the $N = 4$ super Yang-Mills (SYM) theory in four dimensions with gauge group $U(n)$ (see [3] for a physical background) This theory is written in terms of the $N = 1$ SYM theory with three adjoint chiral super-fields X_1, X_2, X_3 coupled by the potential:

$$\Phi_{smooth} = g\mathrm{Tr}([X_1, X_2]X_3)$$

with coupling constant g, where, following the physics literature Tr denotes the map $A \to A_\natural$. From now on we drop Tr, i.e. we denote potentials and their images in $A_\natural$ with the same symbol.

The moduli space of supersymmetric gauge theories is an important and rather well-studied object (a mathematical account of this theory can be found in the recent paper of C. Walton [56]). The marginal deformations, which preserve some conformal symmetry, of the $N = 4$ Superconformal Field Theory have many interesting applications. In particular, within the framework of the AdS/CFT correspondence, they have a nice Supergravity dual descriptions.

If one chooses to preserve $N = 1$ Super Conformal Field Theory then the moduli space of the marginal deformations is given by the potential:

$$\Phi_{\mathrm{marg}} = X_1X_2X_3 - qX_2X_1X_3 + \frac{1}{3}\lambda(X_1^3 + X_2^3 + X_3^3). \tag{6.60}$$

Another important class of deformations is provided by *relevant deformations* which describes the theory away from the Ultra-Violet conformal fixed point:

$$\Phi_{\mathrm{rel}} = m_1X_1^2 + m_2(X_2^2 + X_3^2) + \sum_k d_kX_k.$$

The structure of the vacua of D-brane gauge theories relates to Non-Commutative Geometry via the potentials Φ_{phys} by so called $F - term$ constraints:

$$\frac{\partial\Phi_{phys}}{\partial X_k} = 0, \quad k = 1, 2, 3 \tag{6.61}$$

where $\Phi_{phys} = \Phi_{\mathrm{marg}} + \Phi_{\mathrm{rel}}$. This gives rise to the following non homogeneous relations:

$$\begin{cases} X_1X_2 - qX_2X_1 = & -\Lambda X_3^2 - m_2X_3 - d_3 \\ X_2X_3 - qX_3X_2 = & -\Lambda X_1^2 - m_1X_1 - d_1 \\ X_3X_1 - qX_1X_3 = & -\Lambda X_2^2 - m_2X_2 - d_2 \end{cases} \tag{6.62}$$

This algebra is a particular case of the algebra $A^q_{\Phi_{EG}+\Psi_{EG}}$ studied in subsection 4.4 for $\alpha = \beta = \gamma = -\Lambda$, $a_1 = m_1$, $b_1 = c_1 = m_2$, $a_2 = d_1$, $b_2 = d_2$, $c_2 = d_3$, or in other words, of the general algebra $\mathcal{A}^q$ introduced in subsection 4.8.

Remark 6.1. We precise how this deformation algebra relates to previously studied:

- If $\Lambda = 0$ and $m_1 = m_2 = -\frac{1}{2}$, $e_i = 0$, $i = 1, 2, 3$ then we have the potential of (4.41) and this algebra coincides with the Odesskii degeneration of Sklyanin algebra in subsection 4.5;

- We see that for $\Lambda = 0$, $m_1 = m_2 = \sqrt{q}(q^{-1} - q)$ and $d_i = (1-q)\Omega_i^{(VI)}$ the Poisson algebra (6.62) has the form (1.5).
- If $\Lambda = \frac{c}{a}$ and $q = b/a$, then this Poisson algebra coincides with a deformation of the quadratic Sklyanin Poisson algebra with three generators $q_3(\mathcal{E})$. The latter can be obtained from this deformation by setting the massess to zero: $m_1 = m_2 = 0$.

6.1. **Semi-classical limits.** We now take the semi-classical limit of (6.62) and compare it with the cubic surfaces $\mathcal{M}_\phi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/\langle \phi = 0\rangle)$, for ϕ in table 1. These cubics are endowed with the natural Poisson bracket (2.11). By the correspondence principle

$$\lim_{q\to 1} \frac{[X_1, X_2]}{1-q} = \{x_1, x_2\},$$

and, applying the algebra relations

$$[X_1, X_2] = (q-1)X_2X_1 - \Lambda X_3^2 - m_2X_3 - d_3$$

so that

$$\{x_1, x_2\} = x_1x_2 - \lim_{q\to 1}\frac{\Lambda}{1-q}X_3^2 - \lim_{q\to 1}\frac{m_2X_3}{1-q} + \lim_{q\to 1}\frac{d_3}{1-q},$$

and similarly

$$\{x_2, x_3\} = x_2x_3 - \lim_{q\to 1}\frac{\Lambda}{1-q}X_1^2 - \lim_{q\to 1}\frac{m_1X_1}{1-q} + \lim_{q\to 1}\frac{d_1}{1-q},$$

$$\{x_3, x_1\} = x_1x_3 - \lim_{q\to 1}\frac{\Lambda}{1-q}X_2^2 - \lim_{q\to 1}\frac{m_2X_2}{1-q} + \lim_{q\to 1}\frac{d_2}{1-q}.$$

By a slight abuse of notation, we denote the classical masses again by m_1, m_2, the classical limit of Λ by λ and put $\delta_i = \lim_{q\to 1}\frac{d_i}{1-q}$, so that the Casimir function for this Poisson algebra is

$$\phi_{\mathrm{cl,tot}}(x_1, x_2, x_3) = x_1x_2x_3 - m_1x_1^2 - m_2(x_2^2+x_3^2) - \frac{\lambda}{3}(x_1^3+x_2^3+x_3^3) + \delta_1x_1 + \delta_2x_2 + \delta_3x_3.$$

We see that the corresponding Poisson algebras include all interesting families of quadratic-linear-constant Poisson brackets in $\mathbb{C}[x_1, x_2, x_3]$ and, in particular, for $\lambda = 0$, the family coincides with the Poisson structure on the Painlevé monodromy data cubics. At the same time, by neglecting the terms of degree < 3 in $\phi_{\mathrm{cl,tot}}$ we obtain

$$\phi_{\mathrm{cl,marg}}(x_1, x_2, x_3) = x_1x_2x_3 - m_1x_1^2 - m_2(x_2^2 + x_3^2) - \frac{\lambda}{3}(x_1^3 + x_2^3 + x_3^3), \tag{6.63}$$

that is a perturbation of the classical Sklyanin algebra $q_{3,1}(\mathcal{E})$ (see section 5.1).

6.2. **Degeneration of quadratically perturbed q_3-Sklyanin brackets and Gross-Siebert theta-functions.** Consider the special case of (6.63) with $m_2 = 0$ and $\lambda = \frac{3}{m_1^3}$:

$$\phi_{\mathrm{cl},1}(x_1, x_2, x_3) = x_1x_2x_3 - m_1x_1^2 - \frac{1}{m_1^3}(x_1^3 + x_2^3 + x_3^3). \tag{6.64}$$

This is an example of a central element for the classical Sklyanin algebra perturbed by the quadratic term $m_1x_1^2$.

We introduce the coordinates $y_i, i = 1, 2, 3$ connected to x_1, x_2, x_3 by the following relations:

$$x_1 = \frac{y_1}{\sqrt{m_1}}, \quad x_2 = \frac{y_2}{\sqrt{m_1}}, \quad x_3 = m_1 y_3,$$

so that the Casimir now reads as

$$\phi_{\mathrm{cl},2}(y_1, y_2, y_3) = y_1 y_2 y_3 - y_1^2 - y_3^3 + \frac{(y_1^3 + y_2^3)}{\sqrt{m_1^9}}. \tag{6.65}$$

In the infinite mass limit $m_1 \to \infty$, (6.65) goes evidently to

$$\phi_{\mathrm{cl},3}(y_1, y_2, y_3) = y_1 y_2 y_3 - y_1^2 - y_3^3 \tag{6.66}$$

Note that up to permutations of y_1, y_2, y_3, $\phi_{\mathrm{cl},3}$ is the same as $\tilde{\phi}_{213_0}$, and therefore, as discussed at the end of subsection 5.1.2, the cubic surface $M_{\phi_{\mathrm{cl},3}} \subset \mathbb{C}^3$ given by $\phi_{\mathrm{cl},3}(y_1, y_2, y_3) = y_1 y_2 y_3 - y_1^2 - y_3^3$ can be considered as an affine cone over a singular genus one rational curve $\mathcal{E}_{sing} \subset \mathbb{WP}(3, 1, 2)$. Its coordinate ring

$$\mathbb{C}[M_{\phi_{\mathrm{cl},3}}] = \mathbb{C}[y_1, y_2, y_3]/(y_1 y_2 y_3 - y_1^2 - y_3^3)$$

is isomorphic to the ring of sections $\oplus_{k \geq 0} H^0(\mathcal{E}_{sing}, \mathcal{O}(k))$ of a degree 1 line bundle $\mathcal{O}(1)$) on the nodal rational curve $\mathcal{E}_{sing}$ of arithmetic genus 1 (see [21] ch.5). This cone is parametrised by toric theta-functions ϑ_i, $i = 1, 2, 3$ satisfying the relation

$$\vartheta_1 \vartheta_2 \vartheta_3 = \vartheta_1^2 + \vartheta_3^3$$

(see Theorem 2.34 of [21]).

Now we come back to the Poisson algebra corresponding to (6.64):

$$\{x_1, x_2\} = -\frac{3x_3^2}{m_1^3} + x_1 x_2; \quad \{x_2, x_3\} = -\frac{3x_1^2}{m_1^3} - 2m_1 x_1 + x_2 x_3; \quad \{x_3, x_1\} = \frac{3x_2^2}{m_1^3} + x_3 x_1$$

which will be written in the degenerated coordinates $y_i, i = 1, 2, 3$ as

$$\{y_1, y_2\} = -3y_3^2 + y_1 y_2; \quad \{y_2, y_3\} = -\frac{3y_1^2}{\sqrt{m_1^9}} - 2y_1 + y_2 y_3; \quad \{y_3, y_1\} = \frac{3y_2^2}{\sqrt{m_1^9}} + y_3 y_1. \tag{6.67}$$

From this, in the infinite mass limit we obtain once again (compare with (5.51)) a perturbed cluster Poisson structure:

$$\{y_1, y_2\} = -3y_3^2 + y_1 y_2; \quad \{y_2, y_3\} = -2y_1 + y_2 y_3; \quad \{y_3, y_1\} = y_3 y_1 \tag{6.68}$$

which defines the Poisson algebra structure on the coordinate ring of the affine cone over the curve $\mathcal{E}_{sing}$.

If, instead, we introduce the coordinates $\tilde{y}_i, i = 1, 2, 3$ connected to x_1, x_2, x_3 by the following relations:

$$x_1 = \frac{\tilde{y}_1}{\sqrt{m_1}}, \quad x_2 = \tilde{y}_2, \quad x_3 = \sqrt{m_1}\tilde{y}_3,$$

the Casimir now reads as

$$\phi_{\mathrm{cl},4}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \tilde{y}_1 \tilde{y}_2 \tilde{y}_3 - \tilde{y}_1^2 - \frac{1}{m_1^3}\left(\frac{\tilde{y}_1^3}{\sqrt{m_1^3}} + \tilde{y}_2^3 + m_1^{3/2}\tilde{y}_3^3\right). \tag{6.69}$$

In the infinite mass limit $m_1 \to \infty$, (6.69) goes evidently to

$$\phi_{\mathrm{cl},5}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \tilde{y}_1 \tilde{y}_2 \tilde{y}_3 - \tilde{y}_1^2 \tag{6.70}$$

Note that (up to the change of variable $\tilde{y}_1 = y_1$, $\tilde{y}_2 = y_3$, $\tilde{y}_3 = y_2$) $\phi_{\mathrm{cl},5}$ is the same as $\tilde{\phi}_{112_0}$, and therefore, as before, the cubic surface $M_{\phi_{\mathrm{cl},4}} \subset \mathbb{C}^3$ is an affine cone over the singular curve $\tilde{\mathcal{E}}_{sing} \subset \mathbb{WP}(2,1,1)$. Its coordinate ring

$$\mathbb{C}[M_{\phi_{\mathrm{cl},4}}] = \mathbb{C}[\tilde{y}_1, \tilde{y}_2, \tilde{y}_3]/(\tilde{y}_1\tilde{y}_2\tilde{y}_3 - \tilde{y}_1^2)$$

is isomorphic to the ring of sections $\oplus_{k\geq 0} H^0(\tilde{\mathcal{E}}_{sing}, \mathcal{O}(k))$ of a degree 2 line bundle $\mathcal{O}(1)$) on the degenerated curve $\tilde{\mathcal{E}}_{sing}$ which is a union of conic and a line. This cone is parametrised by Gross-Siebert toric theta-functions ϑ_i, $i = 1, 2, 3$ satisfying the relation

$$(\vartheta_1\vartheta_2 - \vartheta_3)\vartheta_3 = 0$$

(see Proposition 40 of [6]).

By writing the Poisson algebra corresponding to (6.64) in the new coordinates $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ and taking the infinite mass limit we obtain once again (compare with 5.51) a perturbed cluster Poisson structure:

$$\{\tilde{y}_1, \tilde{y}_2\} = \tilde{y}_1\tilde{y}_2; \quad \{\tilde{y}_2, \tilde{y}_3\} = -2\tilde{y}_1 + \tilde{y}_2\tilde{y}_3; \quad \{\tilde{y}_3, \tilde{y}_1\} = \tilde{y}_3\tilde{y}_1 \tag{6.71}$$

which defines the Poisson algebra structure on the coordinate ring of the affine cone over the curve $\tilde{\mathcal{E}}_{sing}$.

6.3. **Quantisation of Gross-Siebert theta functions.** In [6], P. Bousseau proposed a deformation quantisation for some Poisson algebra structures connected with mirror duals of Looijenga pairs (Y, D) where Y is a smooth projective surface and D some singular anticanonical divisor. As examples he considered the deformation quantisation of function algebras on affine varieties $\mathbb{V}_r$ where r is the number of irreducible components of the cubic divisor D. When $r = 1$ the variety $\mathbb{V}_1$ is exactly the affine cone of the nodal curve embedded in the weighted projective space $\mathbb{W}P_{2,1,3}$:

$$\mathbb{V}_1 = M_{\tilde{\phi}_{2,1,3_0}} = \mathrm{Spec} A_{\tilde{\phi}_{2,1,3_0}}$$

where $A_{\tilde{\phi}_{2,1,3}}$ is given in (5.52). The Poisson algebra on $\mathbb{V}_1$ is given by the brackets (5.51).

The Proposition 41 in [6] states that the relations

$$\begin{cases} \sqrt{\hat{q}}\hat{Y}_3\hat{Y}_1 - \frac{1}{\sqrt{\hat{q}}}\hat{Y}_1\hat{Y}_3 = 0 \\ \sqrt{\hat{q}}\hat{Y}_2\hat{Y}_3 - \frac{1}{\sqrt{\hat{q}}}\hat{Y}_3\hat{Y}_2 = (\hat{q} - \hat{q}^{-1})\hat{Y}_1 \\ \sqrt{\hat{q}}\hat{Y}_1\hat{Y}_2 - \frac{1}{\sqrt{\hat{q}}}\hat{Y}_2\hat{Y}_1 = (\hat{q}^{3/2} - \hat{q}^{-3/2})\hat{Y}_3^2 \end{cases}$$

and the central element

$$\hat{\Omega}_{2,1,3}(\hat{Y}) = \hat{Y}_2\hat{Y}_3\hat{Y}_1 - \hat{q}^{1/2}\hat{Y}_1^2 - \hat{q}\hat{Y}_3^3.$$

give the quantisation of (5.51).

In the same paper, Bousseau considered also a deformation quantisation of the function algebra on $\mathbb{V}_2$ related to the mirror dual of the Looijenga pair (Y, D) where the divisor has two connected components, namely for $\mathbb{V}_2 = \mathcal{M}_{\phi^0_{1,1,2}}$, where $\phi_{1,1,2_0}$ os given in (5.53) and the Poisson algebra is the Jacobian algebra on $\mathbb{C}[y_1, y_2, y_3]$ with the brackets (5.54).

A natural question posed in [6] is to make a comparison of his deformation quantisations and the scheme of quantisation following the ideas of Etingof-Ginzburg scheme.

In the next theorem we show that these that these two quantisations lead the same algebras in the case $\mathbb{V}_1$ and $\mathbb{V}_2$.

Theorem 6.2. *The deformation quantisations of the affine Poisson structures on $\mathbb{V}_{1,2}$ obtained in* [6] *coincide (after a proper rescaling) with the appropriate degenerations of the quantum Sklyanin-Painlevé algebras defined by relations* (1.9).

Proof. We start by observing that the quantum algebra corresponding to (6.70) is a degenerate case of the Calabi-Yau algebra $\mathbb{C}\langle X_1, X_2, X_3\rangle/J_{\Phi_{phys}}$ with the potential (6.61). Indeed, by analogy with the classical case, we introduce the coordinates $Y_i, i = 1, 2, 3$ connected to X_1, X_2, X_3 by the following relations:

$$X_1 = \frac{Y_1}{\sqrt{m_1}}, \quad X_2 = \frac{Y_2}{\sqrt{m_1}}, \quad X_3 = m_1 Y_3,$$

to obtain

$$\begin{aligned}\Phi_{phys} = Y_1Y_2Y_3 - qY_2Y_1Y_3 + \frac{\Lambda}{3}\left(m_1^3Y_3^3 + \frac{Y_1^3+Y_2^3}{\sqrt{m_1^3}}\right) + \\ +\frac{1}{2}Y_1^2 + \frac{m_2}{m_1}Y_2^2 + m_1m_2Y_3^2 + e_1Y_1 + e_2Y_3 + e_3Y_2,\end{aligned} \tag{6.72}$$

which is by our discussion a PBW non-homogeneous deformation of the Koszul generalised Sklyanin algebra. By putting $\Lambda = m_1^{-3}$, $m_2 = 0$ and $e_1 = e_2 = e_3 = 0$, we obtain

$$\Phi_{m_1} = Y_1Y_2Y_3 - qY_2Y_1Y_3 + \frac{1}{3}\left(Y_3^3 + \frac{Y_1^3+Y_2^3}{m_1^3\sqrt{m_1^3}}\right) + \frac{1}{2}Y_1^2,$$

and in the limit $m_1 \to \infty$ we obtain

$$\Phi_\infty(Y) = Y_1Y_2Y_3 - qY_2Y_1Y_3 + \frac{1}{3}Y_3^3 + \frac{1}{2}Y_1^2$$

and the corresponding quantum algebra $\mathbb{C}\langle Y_1, Y_2, Y_3\rangle/J_{\Phi_\infty}$ has relations

$$\begin{cases} Y_3Y_1 - qY_1Y_3 = & 0 \\ Y_2Y_3 - qY_3Y_2 = & Y_1 \\ Y_1Y_2 - qY_2Y_1 = & Y_3^2 \end{cases} \tag{6.73}$$

This algebra has central element

$$\Omega_0^{m_1}(Y) = Y_3Y_2Y_1 + \frac{q}{q^2-1}Y_1^2 + \frac{q^2}{q^3-1}Y_3^3 \tag{6.74}$$

and quantises the coordinate ring of the cone over the nodal rational genus 1 curve or the coordinate ring of the affine surface (6.70). But these are the same as (6.73) and (6.74) by setting

$$q = \frac{1}{\hat{q}}, \qquad Y_1 = \frac{(1=\hat{q})(\hat{q}-1)^3(1+\hat{q}+\hat{q}^2)^2}{\hat{q}^{\frac{1}{4}}}\hat{Y}_1, \quad \hat{Y}_2 = \hat{q}^{\frac{7}{4}}Y_2, \quad \hat{Y}_3 = (1-\hat{q}^2-\hat{q}^3+\hat{q}^5)Y_3.$$

We can degenerate the algebra (6.73) further by rescaling the variables Y_1, Y_2, Y_3 and taking different limits. Namely, setting

$$Y_1 \to \epsilon_1 Y_1, \quad Y_2 \to \epsilon_2 Y_2, \quad Y_3 \to \epsilon_3 Y_3,$$

we obtain

$$\begin{cases} Y_3Y_1 - qY_1Y_3 = & 0 \\ Y_2Y_3 - qY_3Y_2 = & \frac{\epsilon_1}{\epsilon_2\epsilon_3}Y_1 \\ Y_1Y_2 - qY_2Y_1 = & \frac{\epsilon_3^2}{\epsilon_1\epsilon_2}Y_3^2 \end{cases} \tag{6.75}$$

with central element

$$\Omega_0^{m_1}(Y) = Y_3Y_2Y_1 + \frac{q}{q^2-1}\frac{\epsilon_1}{\epsilon_2\epsilon_3}Y_1^2 + \frac{q^2}{q^3-1}\epsilon_3^2\epsilon_1\epsilon_2 Y_3^3 \tag{6.76}$$

Imposing $\epsilon_2 = 1$, $\epsilon_1 = \epsilon_3^2$, ϵ_3, in the limit $\epsilon_3 \to 0$, we obtain

$$\begin{cases} Y_1Y_3 - qY_3Y_1 = & 0 \\ Y_2Y_3 - qY_3Y_2 = & 0 \\ Y_2Y_1 - qY_1Y_2 = & Y_3^2 \end{cases} \tag{6.77}$$

and the central element is given by

$$\Omega^{\infty}(Y) = Y_2Y_3Y_1 + \frac{1}{q^2-1}Y_3^2. \tag{6.78}$$

□

Observe that by choosing different values and limits of $\epsilon_1, \epsilon_2, \epsilon_3$ in (6.75), we can recognise the algebras given by the super-pontentials of non-commutative Painlevé cubics (PIV and PII) to which the next two subsections are dedicated.

6.3.1. *One non-zero mass and Painlevé IV.* We consider the deformation provided by addition a single mass term to Φ_{smooth}. The corresponding potential (4.2 of [3]) reads (up to symmetric group Σ_3-action):

$$\Phi_{1m} = X_1X_2X_3 - qX_2X_1X_3 - \frac{m}{2}X_1^2. \tag{6.79}$$

The corresponding ideal is defined by

$$X_1X_2 - qX_2X_1 = 0; \quad X_2X_3 - qX_3X_2 = mX_1; \quad X_3X_1 - qX_1X_3 = 0 \tag{6.80}$$

Taking the Poisson limit $q \to 1$ one gets the cubic Casimir :

$$\phi_{\mathrm{cl,PIV}}(x_1, x_2, x_3) = x_1x_2x_3 - \frac{m}{2}x_1^2.$$

Once again, to link with some of our Painlevé cubics (in the single mass case it will be the PIV cubic) we need to add the linear terms:

$$\Phi_{1,m} = X_1X_2X_3 - qX_2X_1X_3 - \frac{m}{2}X_1^2 + d_1X_1 + d_2X_2 + d_2X_3. \tag{6.81}$$

Taking $d_2 = d_3$ one gets

$$X_1X_2 - qX_2X_1 = d_2; \quad X_2X_3 - qX_3X_2 = mX_1 + d_1; \quad X_3X_1 - qX_1X_3 = d_2 \tag{6.82}$$

and the cubic Casimir ($q \neq \pm 1$)

$$\Phi_{\mathrm{PIV}} = X_1X_2X_3 - qX_2X_1X_3 - \frac{m}{2}X_1^2 + \frac{1}{1-q}(d_1X_1 + d_2(X_2 + X_3)).$$

corresponds to the *PIV* case in the table of cubics 1.

6.3.2. *NC Painlevé II.* Consider the potential

$$\Phi_{\rm PII} = X_1X_2X_3 - qX_2X_1X_3 + (q-1)(X_1 + \Omega_2X_2 + X_3) \tag{6.83}$$

The algebraic relations corresponding to the Jacobian ideal are

(6.84)

$$X_1X_2 - qX_2X_1 = (q-1); \quad X_2X_3 - qX_3X_2 = (q-1); \quad X_3X_1 - qX_1X_3 = (q-1)\Omega_2$$

and the Poisson limit gives the Casimir cubics for the Miwa-Jimbo Painlevé II cases:

$$\phi_{\rm cl,PII} = -x_1x_2x_3 + x_1 + \omega_2x_2 + x_3$$

The authors of [3] argue that, in the framework of study of "orbifold singularities", one should take the relation on the moduli space

$$x_1x_2x_3 - (e_1^nx_1 + e_2^nx_2 + e_3^nx_3) + 2(e_1e_2e_3)^{n/2}T_n(-\frac{w}{2(e_1e_2e_3)^{1/2}}) = 0, \tag{6.85}$$

where $T_n(w) = \cos(n \arccos w)$ is the n- th Chebyshev polynomial.

Taking $e_1 = e_2 = e_3 = \exp(\frac{i\pi}{n})$ and $n = 1$ (which means $T_1(w) = w$) we have the expression

$$x_1x_2x_3 + x_1 + x_2 + x_3 - w = 0,$$

so the Miwa-Jimbo Painlevé Casimir cubic can be considered as the $n = 1$ member of the family (6.85).

6.4. **Le Bryun -Witten algebras.** Our final remark is that the generalised Sklyanin-Painlevé algebra with potential Φ^γ gives an example of the *conformal* $\mathfrak{sl}_2$- *enveloping algebra* $U_{abc}(\mathfrak{sl}_2)$ ([26]). It corresponds to the choice of the parameters $a = c = q; \quad m_1 = m_2 = 1; \quad e_1 = e_3 = 0, e_2 = -1; -\gamma = b$:

$$\left\{ \begin{array}{ll} X_1X_2 - qX_2X_1 = & X_3 \\ X_2X_3 - qX_3X_2 = & X_1 \\ X_3X_1 - qX_1X_3 = & -\gamma X_2^2 + X_2 + 1 \end{array} \right. \tag{6.86}$$

This algebra corresponds to the generalised Sklyanin with $\beta = \gamma = 0$, $a = b = c = q$, case (2) of Theorem 4.10

The central element is

$$\Omega_{LBW} = (q^2-1)X_3X_2X_1 - \gamma\frac{1+q}{q(1+q+q^2)}X_2^3 + qX_1^2 + \frac{1}{q}X_2^2 + qX_3^2.$$

It was proved by Le Bruyn in ([26]) that the conformal $\mathfrak{sl}_2$ enevloping algebras are *Auslander regular* and have the *Cohen-Macaulay property* as finitely generated (left) filtered rings. We observe now that, following the results of Artin, Tate and Van den Bergh ([1]), one can construct a cubic divisor $C \hookrightarrow \mathbb{P}^2$ for any three-dimensional Auslander-regular algebra and the algebra is defined by the divisor and an automorphism $\sigma : C \to C$.

This divisor is defined by the equation

$$[C] = \det \begin{pmatrix} \gamma X_2 & -qX_1 & X_3 \\ X_1 & 0 & -qX_2 \\ -qX_3 & X_2 & 0 \end{pmatrix} = 0, \tag{6.87}$$

where the determinant is calculated quantically as follows:

$$-\gamma qX_2^3 + (q^3-1)X_1X_2X_3 = 0$$

and defines a conic $(-\gamma qX_2^2 + (q^3-1)X_1X_3 = 0)$ and a line $X_2 = 0$. The automorphism σ is given on the line by $\sigma(X_1 : 0 : X_3) = (X_1 : 0 : qX_3)$ and on the conic

by $\sigma(X_1 : X_2 : X_3) = (qX : X_2 : q^{-1}X_3)$. Thus, we see that if $\gamma \equiv 0, q^3 \neq 1$ then the divisor gives the triangular configuration $X_1X_2X_3 = 0$ and if $q^3 = 1$ then the divisor degeneratres in a triple line.

7. del Pezzo of degree 3 and open problems

In this section, we summarise our results concerning del Pezzo of degree 3 and their quantisation in three tables and highlight some open problems for the future.

Let us start by describing Table 4.

The first column contains a list of double affine Hecke algebras. The elliptic DAHA of type $\widetilde{E}_6$ is due to Rains, [44], while the GDAHA of type $E_6^{(1)}$ is due to [16]. The abbreviation "Deg. GDAHA" corresponds to some Whittaker degenerations of the $E_6^{(1)}$ GDAHA [12, 28], the $\check{C}C_1$ DAHA is due to Cherednik [11, 47, 31], while the abbreviation "Deg. DAHA" correspond to the algebras obtained in [27] by Whittaker degeneration.

The second column is the polynomial ϕ such that $\mathcal{M}_\phi$ is the center of the corresponding (elliptic or generalised or degenerate) DAHA for $q = 1$: for the cases of Elliptic DAHA, this was conjectured in [16], for GDAHA it was proved in [16], for the $\check{C}C_1$ in [32] and all other cases in [27, 28].

As discussed is Section 5, the projective completion $\overline{\mathcal{M}}_\phi$ is a del Pezzo of degree 3 with divisor D_∞ - this is specified in the third column of table 4.

The table is split vertically by a double line - the whole right side of the table is due to H. Sakai [48], and we have used his notation here. Before explaining what this double line represents, let us recall the definition of an *Okamoto pair* (X, Δ): this is a pair (X, Δ) where X is a generalised Halphen surface, namely the blow up of 9 points in $\mathbb{P}^2$ in non generic position, and Δ is a divisor that tells us the position of such 9 points. Note that Δ has the same configuration as a degenerate elliptic curve in the classification by Kodaira-Neron. In other words, the 9 non generic points lie at the intersection between Δ and a generic elliptic curve in $\mathbb{P}^2$. The generalised Halphen surfaces are uniquely determined by their divisor Δ listed in the fifth column.

Some of these divisors have multiple points on them. Starting from the fifth row, at the intersection of lines we always have a multiple point, this is denoted by an empty circle. The order of this point can be calculated by removing from the number 9 the order of all other points. The single bullet points mean simple points, the bullets with a circle and a number next to them mean multiple points with the order specified by the number. In the last column we show the blow up of such divisor Δ at the multiple points.

Sakai labels the generalised Halphen surface according to the affine Weyl group corresponding to the intersection matrix of the divisor. Note that in the case $A_0^{(1)}$ Sakai uses two notations according to the divisor, no star means Δ is a smooth elliptic curve, one star means Δ is rational curve with a node. These labels are given in the fourth column.

The first line of the table corresponds to the elliptic Painlevé equation, the next three lines to the multiplicative or q-difference Painlevé equations and the last eight rows correspond to the Painlevé differential equations. There are also additive difference Painlevé equations, which we give in Table 6 because the corresponding quantum algebra is not Calabi-Yau [30]. Finally, there are also high dimension

multiplicative difference Painlevé equations the quantum description of which is postponed [28].

In the case of the Painlevé differential equations, the left and right sides of the table are related by the so-called Riemann-Hilbert correspondence - this was proved by several authors, a nice unified approach can be found in [54]. The basic idea is that the Okamoto pair corresponds to the space of initial conditions of the given equation, while the Looijenga pair corresponds to the monodromy manifold.

Okamoto's theory of initial value spaces [37], developed by Sakai [48], provides a beautiful unification of differential and discrete equations. Whether differential or discrete, initial values for any nonlinear equation, can be regular (meaning the solution will be analytic around the initial point) or can be unbounded (reflecting the existence of a singularity at the initial point). Okamoto compactified this space to the complex projective plane and showed that any subsequent indeterminacy can be removed by resolving the base points through blowup techniques from algebraic geometry. It is a miraculous fact that nine blowups leads to a regularisation of the whole space for all differential and discrete Painlevé equations. For discrete Painlevé equations, there is no satisfactory concept of monodromy manifold - it is true that each additive discrete Painlevé equation comes from the Backlünd transformations of one of the differential ones, so that one could use the monodromy manifold associated to the latter, however without a direct isomonodromic approach, interesting dynamical behaviour may be lost. Moreover for the multiplicative discrete Painlevé equations, a notion of monodromy manifold is completely missing. This leads us to

Conjecture 7.1. For the elliptic and multiplicative/additive discrete Painlevé equations, the Riemann Hilbert correspondence assigns to the generalised Halphen surface in Table 4 the corresponding Looijenga pair.

Intuitively speaking, evidence for this conjecture is provided by the fact that the polynomials defining the divisors D_∞ in the first four lines of Table 4 are the same as those defining the corresponding Halphen divisors Δ - i.e. $D_\infty = \Delta$ for the first four lines in the table.

We list the quantum results in Table 5 and 6. All the quantum algebras in Table 5 are specialisations of the generalised Sklyanin-Painlevé algebra introduced in subsection 4.8.

We conclude by mentioning the relation between the quantum algebras in Table 5 and the matrix generalisations of the Painlevé equations. Building upon work by Retakh and the third author [45], in [5] a set of non-commutative relations which are non-commutative analogues of monodromy data relations for the Painlevé II equation was constructed. The interesting feature of these non-commutative relations is that by taking the scalar degeneration of the non-commutative operator q, one obtains our quantum Painlevé II monodromy variety. This observation opens the possibility of relating higher rank Elliptic/Generalised DAHA to the theory of matrix Painlevé equations.

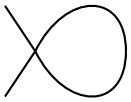
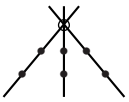
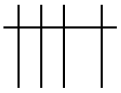
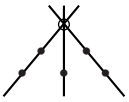
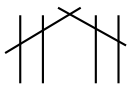
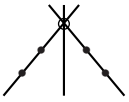
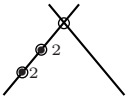
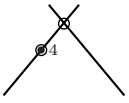
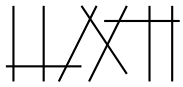
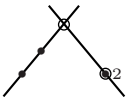
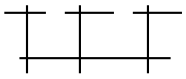
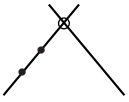
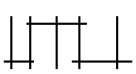
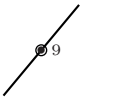
DAHA	Center ϕ for $q=1$	del Pezzo divisor D_∞	Gen. Halphen surface	Halphen divisor Δ	Blow up
Elliptic $\widetilde{E}_6$	$x_1x_2x_3+x_1^3+x_2^3+x_3^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+$ $+x_1^3+x_2^3+x_3^3$	$A_0^{(1)}$		N.A.
GDAHA $E_6^{(1)}$	$x_1x_2x_3+x_1^3+x_2^3+$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+$ $+x_1^3+x_2^3$	$A_0^{(1)\star}$		N.A.
Deg. GDAHA	$x_1x_2x_3+x_1^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+x_1^3$	$A_1^{(1)}$		N.A.
	$x_1x_2x_3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3$	$A_2^{(1)}$		N.A.
DAHA $\check{C}C_1$	$x_1x_2x_3-x_1^2-x_2^2-x_3^2+$ $+\omega_1x_1+\omega_2x_2+\omega_3x_3+\omega_4$	$x_1x_2x_3$	$D_4^{(1)}$		
Deg. DAHA	$x_1x_2x_3-x_1^2-x_2^2+$ $\omega_1x_1+\omega_2x_2+\omega_3x_3+\omega_4$	$x_1x_2x_3$	$D_5^{(1)}$		
	$x_1x_2x_3-x_1^2-x_2^2+$ $\omega_1x_1+\omega_2x_2+\omega_4$	$x_1x_2x_3$	$D_6^{(1)}$		
	$x_1x_2x_3-x_1^2-x_2^2+$ $+\omega_1x_1-x_2$	$x_1x_2x_3$	$D_7^{(1)}$		
	$x_1x_2x_3-x_1^2-x_2^2-x_2$	$x_1x_2x_3$	$D_8^{(1)}$		
	$x_1x_2x_3-x_1^2+$ $\omega_1x_1+\omega_2x_2+\omega_3x_3+\omega_4$	$x_1x_2x_3$	$E_6^{(1)}$		
	$x_1x_2x_3-x_1^2+$ $+\omega_1x_1-x_2-1$	$x_1x_2x_3$	$E_7^{(1)}$		
	$x_1x_2x_3-x_1-x_2+1$	$x_1x_2x_3$	$E_8^{(1)}$		

TABLE 4. Results for del Pezzo of degree 3.

Polynomial ϕ	Quantum relations	potential	Central element
$x_1x_2x_3+x_1^3+x_2^3+x_3^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	(4.31)	$\Phi_{EG}+\Psi_{EG}$ see (4.29), (4.30)	(4.33)
$x_1x_2x_3+x_1^3+x_2^3+$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	(4.38) with $\gamma=0$	$\Phi_{\alpha,\beta,0}+\Psi_{EG}$ see (4.37), (4.30)	(4.39)
$x_1x_2x_3+x_1^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	(4.38) with $\beta=\gamma=0$	$\Phi_{\alpha,0,0}+\Psi_{EG}$ see (4.37), (4.30)	(4.39) with $\beta=0$
$x_1x_2x_3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	(4.31) with $t=0$	$\Phi_{EG}+\Psi_{EG}$ with $t=0$ see (4.29), (4.30)	(4.35)
$\phi_P^{(d)}$, $d=PVI,\dots,PI$ see (2.13)	(1.5) with $\epsilon_i^{(d)},\Omega_i^{(d)}$ in (2.14) (2.15)	$\Phi_{\mathcal{UZ}}$ see (1.7)	(1.8)

TABLE 5. Quantum counterpart of Table 4 (we have squashed the last eight lines of Table 4 into one).

Polynomial ϕ	Quantum relations	Halphen surface	Divisor Δ
$x_1^3-x_2^2x_3$	$x_1^2=x_2^2=0$ $x_3x_2+x_2x_3=0$	$A_0^{(1)^{**}}$	
$x_2^2x_3-x_1^2x_2$	$x_2x_3+x_3x_2-x_1^2=0$ $x_2^2=x_2x_1+x_1x_2=0$	$A_1^{(1)^*}$	
$x_1^3+x_3^3$	$x_1^2=x_2^2=0$	$A_2^{(1)^*}$	

TABLE 6. Non Calabi-Yau cases

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