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▶ **To cite this version:**

Jean-Philippe Monnier. Very special divisors on 4-gonal real algebraic curves. 2013. hal-00808504

HAL Id: hal-00808504
https://hal.science/hal-00808504

Preprint submitted on 5 Apr 2013

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Very special divisors on 4-gonal real algebraic curves

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Mathematics subject classification (2000): 14C20, 14H51, 14P25, 14P99

Abstract

Given a real curve, we study special linear systems called "very special" for which the dimension does not satisfy a Clifford type inequality. We classify all these very special linear systems when the gonality of the curve is small.

1 Introduction and preliminaries

In this note, a real algebraic curve X is a smooth proper geometrically integral scheme over $\mathbb{R}$ of dimension 1. A closed point P of X will be called a real point if the residue field at P is $\mathbb{R}$, and a non-real point if the residue field at P is $\mathbb{C}$. The set of real points $X(\mathbb{R})$ of X decomposes into finitely many connected components, whose number will be denoted by s. By Harnack's Theorem ([BCR, Th. 11.6.2 p. 245]) we know that $s \leq g+1$, where g is the genus of X. A curve with $g+1-k$ real connected components is called an $(M-k)$-curve. Another topological invariant associated to X is $a(X)$, the number of connected components of $X(\mathbb{C}) \setminus X(\mathbb{R})$ counted modulo 2. The pair $(s, a(X))$ is called the topological type of X. If $a(X) = 0$ then $s = g + 1 \mod 2$ (see [K]) and X is called a separating curve.

We will denote by $X_{\mathbb{C}}$ the base extension of X to $\mathbb{C}$. The group $\mathrm{Div}(X_{\mathbb{C}})$ of divisors on $X_{\mathbb{C}}$ is the free abelian group on the closed points of $X_{\mathbb{C}}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the complex variety $X_{\mathbb{C}}$ and also on $\mathrm{Div}(X_{\mathbb{C}})$. We will always indicate this action by a bar. Identifying $\mathrm{Div}(X)$ and $\mathrm{Div}(X_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$, if P is a non-real point of X then $P = Q + \bar{Q}$ with Q a closed point of $X_{\mathbb{C}}$. The group $\mathrm{Div}(X)$ of divisors on X is then the free abelian group generated by the closed points of X. If D is a divisor on X, we will denote by $\mathcal{O}(D)$ its associated invertible sheaf. The dimension of the space of global sections of this sheaf will be denoted by $h^0(D)$. Since a principal divisor has an even degree on each connected component of $X(\mathbb{R})$ (e.g. [10] Lem. 4.1), the number $\delta(D)$ (resp. $\beta(D)$) of connected components C of $X(\mathbb{R})$ such that the degree of the restriction of D to C is odd (resp even) is an invariant of the linear system $|D|$ associated to D. If $h^0(D) > 0$, the dimension of the linear system $|D|$ is $\dim |D| = h^0(D) - 1$. Let K be the canonical divisor. If $h^0(K - D) = \dim H^1(X, \mathcal{O}(D)) > 0$, D is said to be special. If not, D is said to be non-special. By Riemann-Roch, if $\deg(D) > 2g - 2$ then D is non-special. Assume D is effective of degree d. If D is non-special then the dimension of the linear system $|D|$ is given by Riemann-Roch. If D is special, then the dimension of the linear system $|D|$ satisfies

$$\dim |D| \leq \frac{1}{2} d.$$

This is the well known Clifford inequality for complex curves that works also for real curves. The reader is referred to [1] and [11] for more details on special divisors. Concerning real curves, the reader may consult [10].

Huisman ([12, Th. 3.2]) has shown that:

Theorem 1.1 *Assume X is an M-curve (i.e. $s = g + 1$) or an $(M-1)$-curve (i.e. $s = g$). Let $D \in \mathrm{Div}(X)$ be an effective and special divisor of degree d. Then*

$$\dim |D| \leq \frac{1}{2}(d - \delta(D)).$$

Huisman inequality is not valid for all real curves and we have the following theorem.

Theorem 1.2 *[Mo1, Th. A] Let D be an effective and special divisor of degree d. Then either*

$$\dim |D| \leq \frac{1}{2}(d - \delta(D)) \tag{Clif1}$$

or

$$\dim |D| \leq \frac{1}{2}(d - \beta(D)) \tag{Clif2}$$

Moreover, D satisfies the first inequality if either $s \leq 1$ or $s \geq g$.

In this note we are interested in special divisors that do not satisfy the inequality (Clif1) given by Huisman.

Definition 1.3 *Let D be an effective and special divisor of degree d. We say that D is a very special divisor (or $|D|$ is a very special linear system) if D does not satisfy the inequality (Clif 1) i.e. $\dim |D| > \frac{1}{2}(d - \delta(D))$. If D is very special then there exists $k \in \mathbb{N}$ such that*

$$\dim |D| = \frac{1}{2}(d - \delta(D)) + k + 1$$

and k is called the index of D denoted by $\mathrm{ind}(D)$.

We can reformulate Theorem 1.2 with the concept of very special divisors.

Theorem 1.4 *[Mo1, Th. B] Let D be an effective and very special divisor of degree d. Then*

$$\dim |D| \leq \frac{1}{2}(d - \frac{1}{2}(s-2)) \tag{$*$}.$$

In the previous cited paper, a result is obtained in this direction.

Theorem 1.5 *[Mo1, Th. 2.18] Let D be a very special and effective divisor of degree d on a real curve X such that (*) is an equality i.e.*

$$r = \dim |D| = \frac{1}{2}(d - \frac{1}{2}(s-2))$$

then X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd.

Let $D \in \mathrm{Div}(X)$ be a divisor with the property that $\mathcal{O}(D)$ has at least one nonzero global section. The linear system $|D|$ is called base point free if $h^0(D - P) \neq h^0(D)$ for all closed points P of X. If not, we may write $|D| = E + |D'|$ with E a non zero effective divisor called the base divisor of $|D|$, and with $|D'|$ base point free. A closed

point P of X is called a base point of $|D|$ if P belongs to the support of the base divisor of $|D|$. We note that

$$\dim |D| = \dim |D'|.$$

As usual, a g^r_d is an r-dimensional complete linear system of degree d on X. Let $|D|$ be a base point free g^r_d on X. The linear system $|D|$ defines a morphism $\varphi_{|D|} : X \to \mathbb{P}^r_{\mathbb{R}}$ onto a non-degenerate (but maybe singular) curve in $\mathbb{P}^r_{\mathbb{R}}$ i.e. $\varphi_{|D|}(X)$ is not contained in any hyperplane of $\mathbb{P}^r_{\mathbb{R}}$. If $\varphi_{|D|}$ is birational (resp. an isomorphism) onto $\varphi_{|D|}(X)$, the g^r_d (or D) is called simple (resp. very ample). Let X' be the normalization of $\varphi_{|D|}(X)$, and assume D is not simple i.e. $|D - P|$ has a base point for any closed point P of X. Thus, the induced morphism $\varphi_{|D|} : X \to X'$ is a non-trivial covering map of degree $t \geq 2$. In particular, there exists $D' \in \mathrm{Div}(X')$ such that $|D'|$ is a $g^r_{\frac{d}{t}}$ and such that $D = \varphi^*_{|D|}(D')$, i.e. $|D|$ is induced by X'. If g' denote the genus of X', $|D|$ is classically called compounded of an involution of order t and genus g'. In the case $g' > 0$, we speak of an irrational involution on X.

Concerning non-simple very special divisors, a complete description is given in [Mo2]:

Theorem 1.6 *[Mo2, Thm. 2.5, Thm. 4.1] Let D be a very special divisor of degree d such that the base point free part of $|D|$ is non-simple. Then*

(i) *D is base point free,*

(ii) *$\delta(D) = s$,*

(iii) *the index of D is null.*

If moreover $\dim |D| = r > 1$ then the morphism $\varphi_{|D|} : X \to X'$ is a non-trivial covering map of degree 2 and $D = \varphi^*_{|D|}(D')$ with $D' \in \mathrm{Div}(X')$ such that $|D'| = g^r_{\frac{d}{2}}$. Let g' denote the genus of X' and let s' be the number of connected components of $X'(\mathbb{R})$, we have the following additional properties:

(iv) *D' is a base point free non-special divisor and X' is an M-curve.*

(v) *s is even, $s' = \frac{s}{2}$, r is odd and $\delta(D') = s'$.*

(vi) *$a(X) = 0$ and g is odd and there is a very special pencil on X.*

In this note, we give conditions under which a real curve with few real connected components can have a very special system.

Theorem 1.7 *Let X be real curve with $s \leq 4$. If X has a very special linear system then $X_{\mathbb{C}}$ and X are s-gonal, $s \geq 2$, X has a very special pencil and X is a separating curve i.e. $a(X) = 0$.*

If $s \geq g - 4$, we prove that the existence of a very special linear system implies also the existence of a very special pencil.

By the previous theorem, the existence of a very special linear system on a real curve, with $s \leq 4$, forces the gonality of the curve to be s. The following theorem concerns very special linear series on a real curve with a small gonality.

Theorem 1.8 *Let X be real curve such that X and $X_{\mathbb{C}}$ are both n-gonal with $2 \leq n \leq 4$. If X has a very special linear system then X has a very special pencil and X is a separating curve i.e. $a(X) = 0$. Moreover, if $|D|$ is a very special linear system then $\mathrm{ind}(D) = 0$, $\delta(D) = s$, $|D|$ and $|K - D|$ are base point free.*

In the last section of this note, we improve the result of Theorem 1.4.

Theorem 1.9 *Let $|D|$ be a very special linear system of degree d on a real curve X.*

(i) *We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s-2}{2}),$$

with equality if and only if X is hyperelliptic, the g^1_2 is very special and $s = 2$.

(ii) *Assume X is not hyperelliptic. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s-1}{2}),$$

with equality if and only if X is trigonal, a g^1_3 is very special and $s = 3$.

(iii) *Assume X is not hyperelliptic and not trigonal. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s}{2}),$$

with equality if and only if X is 4-gonal, a g^1_4 is very special and $s = 4$.

(iv) *Assume X has gonality ≥ 5. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s+1}{2}).$$

2 Properties of very special divisors

In this section, we recall and establish some results on very special linear systems.

Let D be a special and effective divisor. The linear system $|D|$ is called primitive if $|D|$ and $|K - D|$ are base point free. If $|D|$ is base point free and F is the base divisor of $|K - D|$ then $|D + F|$ is primitive (it is called the primitive hull of $|D|$) and satisfies $\dim|D + F| = \dim|D| + \deg(F)$.

We recall that if $D \in Div(X)$ then $\delta(D) = \delta(K - D)$. By the previous remark and Riemann-Roch, we get:

Lemma 2.1 *[Mo2, Lem. 2.4] Let D be a very special divisor then $K - D$ is also very special and* $\mathrm{ind}(D) = \mathrm{ind}(K - D)$.

Lemma 2.2 *Let D be an effective divisor. Let F be the base divisor of $|D|$. If D is very special then the base point free part $|D - F|$ of $|D|$ is also very special and* $\mathrm{ind}(D-F) \geq \mathrm{ind}(D)$. *Moreover* $\mathrm{ind}(D-F) = \mathrm{ind}(D)$ *if and only if $F = P_1 + \ldots + P_f$ with the P_i some real points among the $\delta(D)$ real connected components on which the degree of the restriction of D is odd, such that no two of them belong to the same real connected component.*

Proof : Set $d = \deg(D)$ and $k = \mathrm{ind}(D)$.

Suppose that there exists a non-real point $Q + \bar{Q}$ such that $Q + \bar{Q} \leq F$. Then $\dim|D| = \dim|D - Q - \bar{Q}| = \frac{1}{2}(d - \delta(D)) + k + 1 = \frac{1}{2}((d-2) - \delta(D)) + (k+1) + 1$ and $\mathrm{ind}(D - Q - \bar{Q}) = \mathrm{ind}(D) + 1$.

Suupose there are two real points P, P' belonging to the same real connected component, such that $P + P' \leq F$, then as before, $\mathrm{ind}(D - P - P') = \mathrm{ind}(D) + 1$.

Suppose that there exists a real point P belonging to a connected component on which the degree of the restriction of D is even, such that $P \leq F$. Then $\dim|D| = \dim|D-P| = \frac{1}{2}(d-\delta(D))+k+1 = \frac{1}{2}((d-1)-(\delta(D)+1))+(k+1)+1 = \frac{1}{2}(\deg(D-P)-(\delta(D-P))+(k+1)+1$ and $\mathrm{ind}(D-P) = \mathrm{ind}(D)+1$.

Suppose that there exists a real point P belonging to a connected component on which the degree of the restriction of D is odd, is contained in the base divisor of $|D|$. Then $\dim|D| = \dim|D-P| = \frac{1}{2}(d-\delta(D))+k+1 = \frac{1}{2}((d-1)-(\delta(D)-1))+k+1 = \frac{1}{2}(\deg(D-P)-(\delta(D-P))+k+1$ and $\mathrm{ind}(D-P) = \mathrm{ind}(D)$. □

Corollary 2.3 *Let D be very special and assume $|D|$ is base point free then the primitive hull $|D'|$ of $|D|$ is very special and $\mathrm{ind}(D') \geq \mathrm{ind}(D)$. Let $E \in |D'-D|$, then $\mathrm{ind}(D') = \mathrm{ind}(D)$ if and only if $E = P_1 + \ldots + P_e$ with the P_i some real points among the $\delta(D)$ real connected components on which the degree of the restriction of D is odd, such that no two of them belong to the same real connected component.*

Proof : Let E denote the base divisor of $K-D$. Then $K-D$ is very special of index $\mathrm{ind}(D)$ (Lemma 2.1) and $K-D-E$ is also very special of index $\geq \mathrm{ind}(D)$ by Lemma 2.2. By Lemma 2.1, $D' = D+E = K-(K-D-E)$ is very special of index $\geq \mathrm{ind}(D)$.

Assume $\mathrm{ind}(D') = \mathrm{ind}(D)$ then $\mathrm{ind}(K-D) = \mathrm{ind}(K-D-E)$ and by Lemma 2.2 we are done. □

Theorem 2.4 *[Mo2, Thm. 3.6] If D is a very special divisor then $\dim|D| \neq 2$.*

Proposition 2.5 *Let D be a very special divisor of degree d such that $\dim|D| = 1$. Then $D = P_1+\cdots+P_s$ with $P_1, \ldots, P_s$ some real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$ i.e. $d = \delta(D) = s$ and $\mathrm{ind}(D) = 0$. Moreover D is primitive.*

Proof : By [Mo2, Prop. 2.1], we only have to prove $K-D$ is base point free.

Suppose that P is a real base point of $|K-D|$. Then $\dim|D+P| = 2 = \frac{1}{2}((s+1)-(s-1))+1 = \frac{1}{2}(\deg(D+P)-(\delta(D+P))+1$ and $D+P$ is very special, impossible by Theorem 2.4.

Suppose that $Q+\bar{Q}$ is a non-real base point of $|K-D|$. Then for a general choice of a real point P, we get $\dim|D+Q+\bar{Q}-P| = 2 = \frac{1}{2}((s+1)-(s-1))+1 = \frac{1}{2}(\deg(D+Q+\bar{Q}-P)-(\delta(D+Q+\bar{Q}-P))+1$ and $D+Q+\bar{Q}-P$ is very special, impossible by Theorem 2.4. □

Proposition 2.6 *Let D be a very special divisor of degree d. Let $k = \mathrm{ind}(D)$. Choose k distinct real connected components $C_1, \ldots, C_k$ on which the degree of the restriction of D is odd. For $i = 1, \ldots, k$, take $(P_i, R_i) \in C_i \times C_i$. Take a real point Q_j in each real connected component on which the degree of the restriction of D is even. Then*

$$D' = D - \sum_{i=1}^{k}(P_i+R_i) - \sum_{j=1}^{\beta(D)} Q_j$$

is very special. Moreover, $\dim|D'| = \frac{1}{2}(d+\delta(D))-k-s+1$ and $\mathrm{ind}(D') = 0$ if the points P_i, R_i, Q_j are sufficiently general.

Proof : Assume $\dim|D| = r = \frac{1}{2}(d-\delta(D))+k+1$. We have $r \geq k+1$ since D can be chosen effective.

We have $k \leq \frac{\delta(D)}{2}$ by Clifford Theorem.

Let $D_1 = D - \sum_{i=1}^{k} P_i$. Then $\delta(D_1) = \delta(D) - k$ and $\dim|D_1| \geq r - k = \frac{1}{2}(d - \delta(D)) + 1 - k = \frac{1}{2}(\deg(D_1) - \delta(D_1)) + 1 \geq 1$. We see that D_1 is very special and $\mathrm{ind}(D_1) = 0$ if the points P_i are general.

If $\beta(D_1) = 0$, the proof is done since D_1 is very special. Assume $\beta(D_1) > 0$. Let P be a real point such that P belongs to a connected component on which the degree of the restriction of D_1 is even. Then $\dim|D_1 - P| \geq \dim|D_1| - 1 \geq \frac{1}{2}(\deg(D_1) - \delta(D_1)) = \frac{1}{2}(\deg(D_1 - P) - \delta(D_1 - P)) + 1$. Since $h^0(D_1 - P) \geq \dim|D_1| > 0$ then $D_1 - P$ is linearly equivalent to an effective divisor and therefore $\deg(D_1 - P) \geq \delta(D_1 - P)$ and $\dim|D_1 - P| \geq 1$ and $D_1 - P$ is very special. In the case P is general, we get $\mathrm{ind}(D_1 - P) = \mathrm{ind}(D_1)$. By repeating the same reasoning for $D_1 - P$, we prove the proposition. □

The following lemma was proved differently in [Mo2, Lem. 3.3].

Lemma 2.7 *Let D be a very special divisor of degree d and index k. Then*

$$d + \delta(D) \geq 2s + 2k.$$

Proof: By Proposition 2.6, the divisor $D' = D - \sum_{i=1}^{k}(P_i + R_i) - \sum_{j=1}^{\beta(D)} Q_j$ is very special and therefore $\dim|D'| \geq 1$. Choosing the points P_i, R_i, Q_j sufficiently general, we have $\dim|D| = 2k + \beta(D) + \dim|D'| \geq 2k + s - \delta(D) + 1$. We obtain

$$\frac{1}{2}(d - \delta(D)) + k + 1 \geq 2k + s - \delta(D) + 1$$

i.e.

$$d + \delta(D) \geq 2s + 2k.$$

□

Corollary 2.8 *Let D be a very special divisor of degree d and index k. We have $d + \delta(D) = 2s + 2k$ if and only if $|D|$ is a pencil.*

Proof: Assume $d + \delta(D) = 2s + 2k$. The linear system $|D'|$ of Proposition 2.6 is a very special pencil. Hence $|D'|$ is primitive (Proposition 2.5) and therefore $D' = D$.

The converse follows easily from Proposition 2.5. □

We improve the result of Lemma 2.7.

Proposition 2.9 *Let D be a very special divisor of degree d and index k such that $|D|$ is not a pencil. Then*

$$d + \delta(D) \geq 2s + 2k + 4.$$

The very special divisor D' constructed from D in Proposition 2.6 satisfies

$$\dim|D'| \geq 3.$$

Moreover $d + \delta(D) = 2s + 2k + 4$ if and only if $\dim|D'| = 3$.

Proof: Choosing the points P_i, R_i, Q_j sufficiently general in Proposition 2.6, the linear system $|D'|$ is very special and $\dim|D'| = \frac{1}{2}(d + \delta(D)) - k - s + 1$. Note that $|D'|$ is not primitive if $\delta(D) < s$ or $k > 0$. By Lemma 2.9, we have $d + \delta(D) \geq 2s + 2k + 2$. If $d + \delta(D) = 2s + 2k + 2$ then $\dim|D'| = 2$, impossible by Theorem 2.4. Therefore $\dim|D'| \geq 3$ and we have $\dim|D| = 2k + \beta(D) + \dim|D'| \geq 2k + s - \delta(D) + 3$. We obtain

$$\frac{1}{2}(d - \delta(D)) + k + 1 \geq 2k + s - \delta(D) + 3$$

i.e.

$$d + \delta(D) \geq 2s + 2k + 4.$$

□

Proposition 2.10 *Let X be an $(M-2)$-curve or an $(M-4)$-curve. If X has a very special linear system then it is a very special pencil g^1_s or its residual $K - g^1_s$. If $s = g - 1$ then the residual of a very special pencil is also a very special pencil.*

Proof : Let D be a very special divisor of degree d and index k on X. By Lemma 2.1, we may assume $d \leq g-1$ and then $d + \delta(D) \leq 2g - 2$ (resp. $\leq 2g - 4$) if $s = g-1$ (resp. $s = g-3$). If $|D|$ is not a pencil then $d+\delta(D) \geq 2g-2+2k+4$ (resp. $\geq 2g-6+2k+4$) if $s = g-1$ (resp. $s = g-3$), impossible. Hence $|D|$ is a very special pencil and by Riemann-Roch, $K-D$ is also a very special pencil in the case $s = g-1$. □

Remark 2.11 *The very special pencils give separating morphisms $X \to \mathbb{P}^1$ in the sense of Coppens [Co].*

Proposition 2.12 *Let X be an $(M-3)$-curve or an $(M-5)$-curve. Then X has no very special linear systems.*

Proof : Let D be a very special divisor of degree d and index k on X. By Lemma 2.1, we may assume $d \leq g-1$ and then $d+\delta(D) \leq 2g-3$ (resp. $\leq 2g-5$) if $s = g-2$ (resp. $s = g-4$). If $|D|$ is not a pencil then $d + \delta(D) \geq 2g - 4 + 2k + 4$ (resp. $\geq 2g - 8 + 2k + 4$) if $s = g - 2$ (resp. $s = g - 4$), impossible. Hence $|D|$ is a very special pencil and then $a(X) = 0$, impossible. □

From the above results we get:

Theorem 2.13 *Let X be a real curve such that $s \geq g - 4$. If X has a very special linear system then X has a very special pencil.*

3 Very special webs

Proposition 3.1 *Let D be a very special divisor of degree d and index k such that $\dim|D| = 3$. Then $|D|$ is base point free, $d = s + 4$, $k = 0$ and $\delta(D) = s$.*

Proof : We have $D' = D$ in Proposition 2.6 and thus $d + \delta(D) = 2s + 2k + 4$. Since $D = D'$ then we have $k = 0$ and $\delta(D) = s$. Therefore, we obtain $d = s + 4$. By Proposition 2.2, the base point free part of $|D|$ is also very special of dimension 3, hence its degree is $s + 4$ and thus D is base point free. □

The following proposition is important in the sequel. We give new examples of very special simple linear systems. This proposition was inspired by M. Coppens.

Proposition 3.2 *Let $|D|$ be a very special simple linear system of degree d such that $\dim|D| = 3$ and $X' = \varphi_{|D|}(X) \subset \mathbb{P}^3$ is contained in an irreducible real quadric surface Q. Then*

(i) *The rank of Q is 4 and $Q \simeq \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$.*

(ii) *X' is a curve of bi-degree $(s, 4)$ on Q.*

(iii) $|D| = g^1_s + g^1_4$, g^1_s *and* g^1_4 *are the pull-backs of the linear pencils on* X' *cut out by the rulings of* Q.

(iv) g^1_s *is a very special pencil and* $\delta(g^1_4) = 0$.

(v) $s \geq 3$ *and in the case* $s = 3$ *then* X' *is smooth and* $|D| = K - g^1_3$.

(vi) *If* $s = 4$ *and* $d \leq g - 1$ *then* X' *is smooth.*

Proof : By Proposition 3.1, the degree d of D is $s + 4$, $\delta(D) = s$ and $\mathrm{ind}(D) = 0$.

If the rank of Q is 3, then $|D| = |2F|$ where $|F|$ is the pencil induced by the ruling of Q. This case is not possible since $\delta(D) \neq 0$ by Clifford Theorem.

Thus Q is smooth. Assume $Q(\mathbb{R}) \simeq S^2$. The rulings of Q induce complex and conjugated pencils $|F|$ and $|\bar{F}|$ on X such that $|D| = |F + \bar{F}|$; this is again impossible since $\delta(D) \neq 0$.

We have $Q \simeq \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ and $|D| = g^1_a + g^1_b$ where g^1_a and g^1_b are induced by the real rulings of Q. The hyperplane section H giving the embedding $Q \hookrightarrow \mathbb{P}^3$ is of bi-degree $(1, 1)$ on Q and then $X' = \varphi_{|D|}(X) \subset \mathbb{P}^3$ is a curve of bi-degree (a, b) on Q. Moreover, X and X' are birational since $|D|$ is simple. We get

$$a + b = s + 4.$$

We have $H_1(Q(\mathbb{R}), \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$ and the possible types of the image of the connected components of $X(\mathbb{R})$ are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Let a', b' and c' be respectively the number of connected components of type $(1, 0)$, $(0, 1)$ and $(1, 1)$. We have $a' + b' = \delta(D) = s$, $a' + c' = \delta(g^1_a)$ and $b' + c' = \delta(g^1_b)$.

Suppose $a' \leq a - 2$ and $b' \leq b - 2$. Since $a' + b' = s$ and $a + b = s + 4$, we get $a' = a - 2$ and $b' = b - 2$. Since a connected component of type $(1, 0)$ intersect a connected component of type $(0, 1)$, if g denote the genus of X, we get

$$g \leq ab - a - b + 1 - (a - 2)(b - 2)$$

i.e.

$$g \leq a + b - 3 = s + 1,$$

which is impossible by Theorem 1.1 and Proposition 2.10.

So we can assume $a' = a$ i.e. g^1_a is a very special pencil. We have $a = s$ and it follows that $b = 4$ and $b' = c' = 0$. If $s \leq 2$ then $g \leq 3$ by the genus formula and this is again impossible by the propositions 2.10 and 2.12.

Assume $s = 3$. Let μ denote the multiplicity of the singular locus of X'. We have $g = 6 - \mu$ by the genus formula. Since D is special, we have $\dim|D| = 3 > d - g = 1 + \mu$ i.e. $\mu \leq 1$. If $\mu = 1$, then s and g have the same parity, impossible since $a(X) = 0$ (X has a very special pencil). Thus X' is smooth and X is an $(M - 4)$-curve. By Proposition 2.10, $|D|$ is residual to a very special g^1_3. Since X has a simple very special linear system, X cannot be hyperelliptic and X is trigonal with a unique g^1_3 ($g > 4$) such that $|D| = g^1_3 + g^1_4$.

Assume now $s = 4$ and $d = s + 4 = 8 \leq g - 1$. Let μ denote the multiplicity of the singular locus of X'. We have $g = 9 - \mu$ by the genus formula and it follows that $\mu = 0$. □

We study the converse of the previous proposition.

Proposition 3.3 *Let* X *be a smooth curve of bidegree* $(s, 4)$ *on a hyperboloïd* $Q \subset \mathbb{P}^3_{\mathbb{R}}$ *with* $s \geq 3$ *denoting the number of connected components of* $X(\mathbb{R})$ *and such that all the connected components of* $X(\mathbb{R})$ *are of type* $(1, 0)$. *Then the embedding* $X \hookrightarrow \mathbb{P}^3$ *is given by a simple very special linear system* $|D| = g^1_s + g^1_4$.

Proof : The embedding $X \hookrightarrow \mathbb{P}^3$ is clearly given by a simple linear system $|D| = g^1_s + g^1_4$ with $\delta(g^1_s) = s$ and $\delta(g^1_4) = 0$. It remains to show that D is special. Since X is smooth, we have $g = 3s-3$ and thus D is special if $\dim|D| = 3 > d-g = s+4-3s+3$ i.e. $s > 2$. □

Remark 3.4 *The existence of curves of bidegree $(s, 4)$ with prescribed types of Proposition 3.3 for the real connected components is proved by Zvonilov [Zv] for $s = 3$ and $s = 4$.*

4 Curves with a small number of real connected components

We study the existence of very special linear systems on real curves with $s \leq 4$.

The following theorem summarizes all the results proved in this section.

Theorem 4.1 *Let X be real curve with $s \leq 4$. If X has a very special linear system then $X_{\mathbb{C}}$ is s-gonal and X has a very special pencil.*

By Theorem 2.13, the same conclusion can be drawn if $s \geq g - 4$.

An open question is to know if the statement of Theorem 4.1 is correct without any hypothesis on s. If the answer to this question is the affirmative then very special linear series will only exist on separating real curves.

By Theorem 1.2, we only have to consider curves with $2 \leq s \leq 4$.

We will use several times the following proposition:

Proposition 4.2 *Let $|D|$ be a base point free, simple, and very special linear system of degree d and index k such that $d \geq g$. Then*

$$\dim|K - D| \leq \delta(D) - 2k - 2.$$

Proof : Set $r = \dim|D|$, we have $r = \frac{1}{2}(d - \delta(D)) + k + 1$ i.e.

$$d = 2r + \delta(D) - 2k - 2.$$

Since $d \geq g$, $2D$ is non-special and therefore $\dim|2D| = 2d-g = 4r+2\delta(D)-4k-4-g$. Since $|D|$ is simple and base point free, by [1, Ex. B.6, Chap. 3] (a consequence of the uniform position lemma) (note that there is a misprint in the exercise, the correct formula should be $r(\mathcal{D}+\mathcal{E}) \geq r(\mathcal{D}) + 2r(\mathcal{E}) - r(\mathcal{E}-\mathcal{D}) - 1$), we get $\dim|2D| \geq 3r - 1$. Therefore

$$4r + 2\delta(D) - 4k - 4 - g \geq 3r - 1$$

$$r = \frac{1}{2}(d - \delta(D)) + k + 1 \geq 4k - 2\delta(D) + g + 3$$

$$d - \delta(D) + 2k + 2 \geq 8k - 4\delta(D) + 2g + 6$$

$$d \geq 6k - 3\delta(D) + 2g + 4.$$

Hence $\deg(K - D) = 2g - 2 - d \leq 3\delta(D) - 6k - 6$. By Lemma 2.1, we obtain finally

$$\dim|K - D| = \frac{1}{2}(\deg(K - D) - \delta(D)) + k + 1 \leq \delta(D) - 2k - 2.$$

□

Proposition 4.3 *Let X be a real curve such that $s = 2$. If X has a very special linear system then X is hyperelliptic and the g^1_2 is very special i.e. $\delta(g^1_2) = 2$.*

Proof : Assume D is a very special divisor of degree d and index k. Then $\dim|D| = r = \frac{1}{2}(d - \delta(D)) + k + 1 \geq \frac{1}{2}d + k$ since $\delta(D) \leq 2$. By Clifford Theorem we get $k = 0$. Since the null divisor and K are not very special and since we have an equality in Clifford Theorem, it follows that X is hyperelliptic. By [Mo1, Prop. 2.10], we get $\delta(g^1_2) = 2$. □

Proposition 4.4 *Let X be a real curve such that $s = 3$. If X has a very special linear system then $X_{\mathbb{C}}$ is a trigonal curve and there exists a real g^1_3 very special i.e. $\delta(g^1_3) = 3$.*

Proof : Assume D is a very special divisor of degree d and index k. Then $\dim|D| = r = \frac{1}{2}(d - \delta(D)) + k + 1 \geq \frac{1}{2}d + k - \frac{1}{2}$. By Clifford Theorem we get $k = 0$ i.e.

$$r = \frac{1}{2}(d - \delta(D)) + 1.$$

If $\delta(D) \leq 2$ we get a contradiction since in this case X would be hyperelliptic (equality in Clifford inequality) and then $s = 2$ (Proposition 4.3). Therefore

$$\delta(D) = 3,$$

d is odd and

$$r = \frac{1}{2}(d - 1).$$

By Lemma 2.2, since the index of any special divisor is null and since the δ of any very special divisor is equal to 3, we can conclude that any very special linear system is base point free i.e. any very special linear system is primitive.

Assume D is simple and $d \leq g - 1$. Castelnuovo's bound gives $r \leq \frac{1}{3}(d+1)$ [Beau] i.e.

$$\frac{1}{2}(d - 1) \leq \frac{1}{3}(d + 1)$$

$$d \leq 5.$$

Then $r \leq 2$ and it is impossible by Theorem 2.4.

Assume D is simple and $d \geq g$. By Proposition 4.2 we get $\dim|K - D| \leq 1$ and it follows that $|K - D|$ is a very special pencil i.e. a g^1_3 with $\delta(g^1_3) = 3$.

Assume D is non simple. If $|D|$ is a pencil there is nothing to do. If $\dim|D| > 1$, the existence of a very special g^1_3 is given by Theorem 1.6.

If $X_{\mathbb{C}}$ is not trigonal then $X_{\mathbb{C}}$ must be hyperelliptic since the complex gonality is less than the real gonality and since X has a special divisor. Since the g^1_2 of an hyperelliptic curve is unique, X must be hyperelliptic. By [Mo1, Prop. 3.10], $s = 2$, contradiction. □

Lemma 4.5 *Let X be a real curve such that $s = 4$. If D is a very special divisor on X then $\mathrm{ind}(D) = 0$ and one of the following statements holds:*

(i) *$\delta(D) = 3$ and $|D|$ is primitive.*

(ii) *$\delta(D) = 4$ and the base part of $|D|$ is empty or a real point.*

Proof : Assume D is a very special divisor of degree d and index k. Then $\dim|D| = r = \frac{1}{2}(d - \delta(D)) + k + 1 \geq \frac{1}{2}d + k - 1$. We get $k = 0$ since X can not be hyperelliptic [Mo1, Prop. 3.10] (if $k = 1$ we have equality in Clifford inequality, if $k > 1$ we contradict Clifford inequality). Therefore

$$r = \frac{1}{2}(d - \delta(D)) + 1.$$

We have $\delta(D) = 3$ or $\delta(D) = 4$ since X is not hyperelliptic (if $\delta(D) = 2$ we have equality in Clifford inequality, if $\delta(D) \leq 1$ we contradict Clifford inequality).

Assume $\delta(D) = 3$. By Lemma 2.2, since the index of any very special divisor is null and since the δ of any very special divisor is equal to 3 or 4, we can conclude that $|D|$ is base point free. Since $\delta(K - D) = 3$, $|K - D|$ is also base point free i.e. $|D|$ is primitive.

Assume $\delta(D) = 4$. By Lemma 2.2, since the index of any very special divisor is null and since the δ of any very special divisor is equal to 3 or 4, we conclude that the base part of $|D|$ is empty or a real point. □

Theorem 4.6 *Let X be a real curve such that $s = 4$. If X has a very special linear system then $X_{\mathbb{C}}$ is a 4-gonal curve and there exists a very special g^1_4 i.e. $\delta(g^1_4) = 4$. Moreover, if $|D|$ is a very special linear system on X then $|D|$ is primitive,* $\mathrm{ind}(D) = 0$ *and $\delta(D) = 4$.*

Proof : Assume D is a very special divisor of degree d. By Lemma 4.5 we know that $\mathrm{ind}(D) = 0$ and that $\delta(D) \geq 3$.

Suppose first that $\delta(D) = 3$. We know that $|D|$ is primitive (Lemma 4.5) and that $|D|$ and $|K - D|$ are simple (Theorem 1.6). Changing D by $K - D$ if necessary, we may assume that $d \leq g - 1$. We have

$$\dim|D| = r = \frac{1}{2}(d - 3) + 1.$$

By Castelnuovo's bound

$$r \leq \frac{1}{3}(d + 1)$$

and we get $d \leq 5$ and thus $r \leq 2$. Since $|D|$ is simple, it follows from Theorem 2.4 that this case is not possible.

Suppose now that $\delta(D) = 4$. If $|D|$ is not base point free then the base divisor is a real point P by Lemma 4.5, but then $D - P$ is a very special divisor (Lemma 2.2) with $\delta(D - P) = 3$, we have shown previously that it is impossible. Thus $|D|$ and $|K - D|$ are base point free i.e they are primitive.

We have

$$\dim|D| = r = \frac{1}{2}(d - 4) + 1 = \frac{1}{2}d - 1.$$

Assume $|D|$ is simple and $d \leq g - 1$. By Castelnuovo's bound

$$r \leq \frac{1}{3}(d + 1)$$

and we get $d \leq 8$ and $r \leq 3$. By Theorem 2.4 and since $|D|$ is simple we get $r = 3$ and $d = 8$. By [Beau, Lem. 5.1, Rem. 5.2], X is an extremal curve in the sense of Castelnuovo i.e. $|D|$ is very ample and $X \simeq \varphi_{|D|}(X) \subset \mathbb{P}^3$ is a space curve of maximal genus. By [1, p. 118], $\varphi_{|D|}(X)$ lie on a unique quadric surface Q. By Proposition 3.2, X has a very special g^1_4.

Assume $|D|$ is simple and $d \geq g$. By Proposition 4.2, Lemma 2.1 and Theorem 2.4, $|K - D|$ is a very special g^1_4.

Assume $|D|$ is non simple. If $\dim|D| = 1$ there is nothing to do. If $\dim|D| \geq 2$ the existence of a very special g^1_4 follows from Theorem 1.6.

If the gonality of $X_{\mathbb{C}}$ is ≤ 3 then we contradict [Mo1, Prop. 3.10] and Theorem 5.1. □

5 Real trigonal curves

We study the existence of very special linear series on non hyperelliptic curves with a complex g^1_3.

Theorem 5.1 *Let X be a real curve such that $X_{\mathbb{C}}$ is trigonal. Any very special linear system on X is a very special g^1_3 or the residual of a very special g^1_3. In this situation, $s = 3$ and $a(X) = 0$.*

Proof : Remark that since X is not hyperelliptic then $s \geq 3$ (Proposition 4.3).

Assume $g \leq 4$. By Theorem 1.1, Propositions 2.10 and 4.4, if X has a very special system $|D|$ then $g = 4$, $s = 3$ and $|D| = g^1_3$ or $|D| = K - g^1_3$.

Assume $g > 4$. Then $X_{\mathbb{C}}$ has a unique g^1_3 and this g^1_3 must be real. Suppose D is very special of degree $d \leq g - 1$, index k and suppose moreover $|D|$ is primitive. Set $r = \dim|D|$. Using the fact that the Maroni's invariant m of X (it is the first scrollar invariant of the g^1_3) is well understood, we have $\frac{g-4}{3} \leq m \leq \frac{g-2}{3}$, it is proved in [CKM, Example 1.2.7] that $|D| = r.g^1_3$. Since $\delta(g^1_3) = 1$ or 3, we consider these two cases separatly.

• $\delta(g^1_3) = 1$: If r is odd then we get

$$r = \frac{1}{2}(3r - 1) + k + 1$$

i.e. $r < 0$, impossible. If r is even then we get

$$r = \frac{1}{2}(3r) + k + 1$$

i.e. $r < 0$, impossible.

• $\delta(g^1_3) = 3$: If r is odd then we get

$$r = \frac{1}{2}(3r - 3) + k + 1$$

and therefore $r = 1$ i.e. $|D| = g^1_3$. It is easy to see that the case r is even is not possible.

We have proved that any primitive very special linear system on X is the very special g^1_3 or its residual. Since the index of any primitive very special linear system on X is null, it follows from Lemma 2.2 and Corollary 2.3 that the index of any very special linear system on X is null. Suppose now $|D|$ is very special but not primitive, let $|D'|$ denote the primitive hull of the base point free part of $|D|$. By Lemma 2.2 and Corollary 2.3, we must have $\delta(D') < \delta(D)$, impossible since $\delta(D') = s = 3$. □

6 Four-gonal real curves

In this section, we study the existence of very special linear series on four-gonal real curves. We suppose X is a real curve such that $X_{\mathbb{C}}$ is 4-gonal and such that there exist a base point free g^1_4 on X. We do not assume this g^1_4 is unique. In summary, we assume that $X_{\mathbb{C}}$ and X are both 4-gonal.

In this section we will use several times the following Lemma due to Eisenbud [CM, Lem. 1.8]. Note that this lemma concerns linear systems over $\mathbb{C}$ but the proof works also over $\mathbb{R}$.

Lemma 6.1 *Let g^t_n, g^r_m $(t, r \geq 1)$ be real complete linear systems on a real curve X. If $g^t_n + g^r_m$ has the minimum possible dimension $t + r$ then there exists a real base point free pencil g^1_e such that $g^t_n = t.g^1_e$ and $g^r_m = r.g^1_e$.*

Definition 6.2 *A linear system $|D|$ is called "non-trivial" if it is base point free and if $|D|$ and $|K - D|$ have both dimension ≥ 1.*

Remark 6.3 *A base point free very special linear system is always non-trivial.*

Definition 6.4 *We say that a non-trivial linear system $|D|$ of degree d and dimension r is*
• *of type 1 (for the g^1_4) if it is composed of the g^1_4 i.e. $|D| = r.g^1_4$.*
• *of type 2 (for the g^1_4) if the residual of $|D|$ is composed of the g^1_4 i.e. $|K - D| = r'.g^1_4 + F$ with F the base divisor of $|K - D|$ and $r' = \dim|K - D| = g - d + r - 1$.*

Proposition 6.5 *(Very special linear systems of type 1)*
Let $|D|$ be a base point free very special linear system of type 1 for the g^1_4. Then $|D| = g^1_4$ and thus the g^1_4 must be very special. In this situation, $s = 4$ and $a(X) = 0$.

Proof : Let d be the degree of D and let k be the index of D. We have

$$\dim|D| = r = \frac{1}{2}(d - \delta(D)) + k + 1.$$

Since $|D|$ is of type 1, we also have $|D| = r.g^1_4$.

If r is even, then $\delta(D) = 0$, impossible since it will contradict Clifford Theorem.

Assume r is odd, we have $d = 4r$ and $\delta(D) = \delta(g^1_4) = 0$ or 2 or 4.
• If $\delta(g^1_4) = 0$ then we contradict Clifford Theorem.
• If $\delta(g^1_4) = 2$ then $r = \frac{1}{2}(4r - 2) + k + 1$. It follows that $k = 0$ and we have an equality in Clifford inequality, impossible since $X_{\mathbb{C}}$ is 4-gonal.
• If $\delta(g^1_4) = 4$ then $s = 4$ and the g^1_4 is very special. We have $r = \frac{1}{2}(4r - 4) + k + 1$ i.e. $r = 1 - k$ and the proof is done. □

Proposition 6.6 *(Very special linear systems of type 2)*
Let $|D|$ be a base point free very special linear system of type 2 for the g^1_4. Then $|D| = |K - g^1_4|$ and the g^1_4 must be very special. In this situation $s = 4$ and $a(X) = 0$.

Proof : By Lemma 2.1, $K - D$ is very special. By Lemma 2.2, the moving part of $|K - D|$ is very special of type 1. From Proposition 6.5, this moving part is the g^1_4 which is very special. Therefore $|K - D|$ is a very special pencil but then it must be base point free by Theorem 1.6. Thus $|K - D| = g^1_4$ and the proof is done. □

Proposition 6.7 *Let $|D|$ be a base point free very special linear system such that $|D|$ is not a pencil (particularly, $|D|$ is not of type 1) and $|D|$ is not of type 2. Then*

$$\dim|D - g_4^1| = \dim|D| - 2$$

and $|D - g_4^1|$ is base point free.

Proof : Set $r = \dim|D|$ and $d = \deg(D)$. We have

$$r - 4 \leq \dim|D - g_4^1| < r$$

since $|D|$ is base point free. Remark that $r \geq 3$.

Assume $\dim|D - g_4^1| = r - 1$. Then $\dim|D - g_4^1 + g_4^1| = r = \dim|D - g_4^1| + \dim g_4^1$. Let $|E|$ denote the moving part of $|D - g_4^1|$ and let F be the base divisor of $|D - g_4^1|$. We have

$$\dim|E| + \dim g_4^1 \leq \dim|E + g_4^1| \leq \dim|D - g_4^1 + g_4^1| = \dim|E| + \dim g_4^1$$

hence

$$\dim|E| + \dim g_4^1 = \dim|E + g_4^1|.$$

By Lemma 6.1 we get $|E| = (r-1).g_4^1$. Hence $|D| = |r.g_4^1 + F|$ and $F = 0$ since $|D|$ is base point free. Therefore, $|D|$ is a very special linear system of type 1 and by Proposition 6.5, we get $r = 1$, a contradiction.

Assume $\dim|D - g_4^1| = r - 4$. By Riemann-Roch we get $\dim|K - (D - g_4^1)| = \dim|K - D + g_4^1| = \dim|D - g_4^1| - (d-4) + g - 1 = r - 4 - d + 4 + g - 1 = r - d + g - 1 = \dim|K - D|$. It is impossible since $\dim|K - D + g_4^1| \geq \dim|K - D| + 1$.

Assume $\dim|D - g_4^1| = r - 3$. By Riemann-Roch $\dim|K - (D - g_4^1)| = r - d + g$ i.e. $\dim|K - D + g_4^1| = \dim|K - D| + \dim g_4^1$. Let $|E|$ denote the moving part of $|K - D|$ and let F be the base divisor of $|K - D|$. Set $r' = \dim|K - D| = r - d + g - 1$. By Lemma 2.1 and Lemma 2.2, E is very special. We have $\dim|E + g_4^1| \leq \dim|K - D + g_4^1| = \dim|K - D| + 1 = \dim|E| + 1$. We also have $\dim|E + g_4^1| \geq \dim|E| + 1$. By Lemma 6.1, $|E| = r'.g_4^1$ and then $|K - D| = r'.g_4^1 + F$ (F is the fix part). Therefore, $|D|$ is a very special linear system of type 2, a contradiction with the hypotheses.

We prove now that $|D - g_4^1|$ is base point free. Let $|E|$ denote the moving part of $|D - g_4^1|$ and let F be the base divisor of $|D - g_4^1|$. Let e (resp. f) denote the degree of E (resp. F). We have $\dim|E| = r - 2$ and $\dim|E + g_4^1| \geq r - 1$.
Assume $\dim|E + g_4^1| = r - 1$. By Lemma 6.1, $E = (r-2).g_4^1$ and thus $|D| = |(r-1).g_4^1 + F|$. We get

$$r = \frac{1}{2}(4r - 4 + f - \delta(D)) + k + 1$$

i.e.

$$2r = 2 - f + \delta(D) - 2k.$$

If r is odd then $\delta(D) = \delta(F)$ and we get

$$2r = 2 - f + \delta(F) - 2k$$

i.e. $r \leq 1$ since $f \geq \delta(F)$, contradicting the hypotheses. If r is even then $r \geq 4$ by Theorem 2.4. Since $\delta(D) \leq \delta(F) + 4$ then

$$2r \leq 6 - f + \delta(F) - 2k$$

i.e. $r \leq 3$, contradiction.
We have proved that $\dim|E + g_4^1| \geq r$ and, since $\dim|E + g_4^1| \leq \dim|D| = r$, we get $\dim|E + g_4^1| = r$. Therefore F is contained in the base divisor of $|D|$ i.e. $F = 0$. □

Proposition 6.8 *Let $|D|$ be a primitive very special linear system such that $|K-D|$ is not a pencil and $|D|$ is different from the g^1_4. Then*

$$\dim|D+g^1_4| = \dim|D|+2$$

and $|D+g^1_4|$ is also primitive.

Proof : The linear system $|K-D|$ is base point free, very special and it is not a pencil. If $|K-D|$ is of type 2 then $|D|$ is of type 1 and we get $|D| = g^1_4$ by Proposition 6.5, a contradiction with the hypotheses. We may apply Proposition 6.7 for $|K-D|$ and we get

$$\dim|K-D-g^1_4| = \dim|K-D|-2$$

and $|K-(D+g^1_4)|$ is base point free. By Riemann-Roch, we get $\dim|D+g^1_4| = \dim|D|+2$. To finish the proof, we remark that $|D+g^1_4|$ is base point free since $|D|$ and g^1_4 are both base point free. □

Theorem 6.9 *(Very special linear systems on a 4-gonal curve with $\delta(g^1_4)=0$)*
Let X be a real curve with a fixed g^1_4 with $\delta(g^1_4)=0$ and such that $X_{\mathbb{C}}$ is 4-gonal. Let $|D|$ be a very special linear system of dimension r on X then
- *D is primitive.*
- *r is odd,* $\mathrm{ind}(D)=0$ *and $\delta(D)=s$.*
- *$|D| = |\frac{r-1}{2}.g^1_4 + g^1_s|$ with g^1_s a very special pencil.*

Proof : We note that a very special linear system on X can not be of type 1 and can not be of type 2. Let $|D|$ be a base point free very special linear system which is not a pencil. Then $\dim|D-g^1_4| = \dim|D|-2$ by Proposition 6.7 and it is easy to see that $|D-g^1_4|$ is a base point free very special linear system of index $\mathrm{ind}(D)$. If $|D-g^1_4|$ is not a pencil, we continue the same process, and so on, and by Theorem 2.4 it follows that $D-\frac{r-1}{2}g^1_4$ is a very special pencil g^1_s. We also obtain that $\mathrm{ind}(D) = \mathrm{ind}(g^1_s) = 0$ and $\delta(D) = \delta(g^1_s) = s$. Since the index of any base point free very special linear system is null, it follows from Lemma 2.2 that the index of any very special linear system is also null. Since the index is always null and the δ invariant of any base point free very special linear system is equal to s, it follows from Lemma 2.2 that the base divisor of any very special linear system is also null. □

Definition 6.10 *We say that a non-trivial linear system $|D|$ such that* $\dim|D| = r$ *is of type 3 (for the g^1_4) if*

$$|D| = |(r-1).g^1_4 + F|$$

with F effective. Note then that $\dim|F| \leq 1$*, and for $F \neq 0$ we have* $\dim(r-1).g^1_4 = r-1$.

Proposition 6.11 *Let $|D|$ be a base point free very special linear system of type 3. Then $|D|$ is a very special pencil i.e. a g^1_s with $\delta(g^1_s)=s$.*

Proof : Let d be the degree of D and let k be the index of D. We have

$$\dim|D| = r = \frac{1}{2}(d-\delta(D)) + k + 1.$$

Assume first that $F=0$. Since $d = 4r-4$ we get

$$2r = 2 + \delta(D) - 2k.$$

If r is odd then $\delta(D) = 0$ and we get $r = 1 - k$, impossible. If r is even then $r \geq 4$ by Theorem 2.4. Since $\delta(D) \leq 4$ we get $r \leq 3 - k$, again impossible.

Assume now that $F \neq 0$ and let f denote its degree. Since $d = 4r - 4 + f$ we get

$$2r = 2 - f + \delta(D) - 2k.$$

If r is odd then $\delta(D) = \delta(F)$ and we have $2r = 2 - f + \delta(F) - 2k$. Since $f \geq \delta(F)$ it follows that $r = 1$, $\delta(F) = f$ and $|D| = |F|$ is a very special pencil. If r is even then $r \geq 4$ by Theorem 2.4. Since $\delta(D) \leq \delta(F) + 4$ we get $2r \leq 6 + \delta(F) - f - 2k$ i.e. $r \leq 3$, impossible. □

Proposition 6.12 *Let X be a real curve with a fixed g^1_4 with $\delta(g^1_4) = 2$ and such that $X_{\mathbb{C}}$ is 4-gonal. If $|D|$ is a base point free non-simple very special linear system on X then $|D|$ is a pencil.*

Proof: Let $|D|$ be a base point free non-simple very special linear system on X such that $\dim|D| > 1$. By Theorem 1.6, $\varphi_{|D|} : X \to X'$ has degree two and X' is an M-curve of genus $g' = \frac{s}{2} - 1$ and the inverse image of any connected component of $X'(\mathbb{R})$ is a disjoint union of two connected components of $X(\mathbb{R})$. By Propositions 4.3 and 4.4, we have $s \geq 4$. By Theorem 1.1 we get $4 \leq s \leq g - 1$. Assume the g^1_4 is not induced by X' i.e. g^1_4 is not of the form $\varphi^*_{|D|}(g^1_2)$ for a g^1_2 on X'. By [CKM, Cor. 2.2.2], we must have $4 \geq g - 2g' + 1$ and $g' \geq 1$. Since $s = 2g' + 2$ and $g \leq 2g' + 3$ we obtain $s \geq g - 1$. By Theorem 1.1 and Proposition 2.10, we get $s = g - 1$ and $|D|$ is a pencil, impossible. Hence $g^1_4 = \varphi^*_{|D|}(g^1_2)$ for a g^1_2 on X'. Thus $\delta(g^1_4) \neq 2$, a contradiction. □

Lemma 6.13 *Let X be a real curve with a fixed g^1_4 with $\delta(g^1_4) = 2$ and such that $X_{\mathbb{C}}$ is 4-gonal. If $|D|$ is a very special linear system on X then*

$$\dim|D| \neq 3.$$

Proof: Assume $|D|$ is base point free and very special with $\deg(D) = d$ and $\dim|D| = 3$. From Proposition 3.1, it follows that $|D|$ is base point free and that $\delta(D) = s$. By Proposition 6.12, $|D|$ is simple. By Proposition 6.7, $|D| = |g^1_4 + g^1_s|$ with $\delta(g^1_s) = s - 2$. But then $\varphi_{|D|}(X)$ is contained in a quadric surface of $\mathbb{P}^3$ (see [K, Lem. 1.5] for example). By Proposition 3.2, $|D|$ cannot be very special, a contradiction. □

Proposition 6.14 *Let X be a real curve with a fixed g^1_4 with $\delta(g^1_4) = 2$ and such that $X_{\mathbb{C}}$ is 4-gonal. If X has a very special pencil g^1_s then $s = g - 1$ and any very special linear system on X is a pencil.*

Proof: Let g^1_s be a very special pencil on X.

If $\dim|K - g^1_s| = 1$ then we get $s = g - 1$ by Riemann-Roch.

For the rest of the proof, we assume $\dim|K - g^1_s| > 1$. We denote by $|D|$ the base point free linear system $|g^1_s + g^1_4|$. By Lemma 6.8, $\dim|g^1_s + g^1_4| = 3$ and $|D| = |g^1_s + g^1_4|$ is base point free.

Suppose first that $|D|$ is simple. The curve $X' = \varphi_{|D|}(X)$ is birational to X and X' is contained in a quadric surface of $\mathbb{P}^3$. Thus $X' = \varphi_{|D|}(X) \subset \mathbb{P}^3$ is a curve of bi-degree (a, b) on Q and we have

$$a + b = s + 4.$$

Arguing as in the proof of Proposition 3.2, if a', b', c' denote respectively the number of connected components of type $(1,0)$, $(0,1)$ and $(1,1)$ then we have $a'+b' = \delta(D) = s-2$, $a'+c' = \delta(g^1_s) = s$ and $b'+c' = \delta(g^1_4) = 2$. Therefore $a'+b' = a'-b'$ and thus $b' = 0$, $a' = s-2$, $c' = 2$. Since the $s-2$ connected components of type $(1,0)$ intersect each connected component of type $(1,1)$, the genus formula gives

$$g \leq 4s - 4 - s + 1 - 2(s-2) = s+1.$$

By Theorem 1.2, we get $s = g-1$ and the rest of the proof in this case follows from Proposition 2.10.

Suppose now that $|D|$ is not simple. Let $d = s+4$ denote the degree of D. It means that $\varphi_{|D|}$ has some degree ≥ 2 i.e. $\varphi = \varphi_{|D|} : X \to X'$ is a non-trivial covering map of degree $t \geq 2$ on a real curve X' of genus g'. Moreover, there exists $D' \in \mathrm{Div}(X')$ of degree $d' = \frac{d}{t}$ such that $|D'| = g^3_{d'}$ and such that $D = \varphi^*(D')$.

Assume $t \geq 3$. Let $Q'+\bar{Q}'$ be a non-real point of $X'(\mathbb{R})$. Let $D_1 = D - \varphi^*(Q'+\bar{Q}')$ and denote by $d_1 = d-2t$. We may clearly assume $D'-Q'-\bar{Q}'$ effective and $\dim|D_1| = 1$. Since $Q'+\bar{Q}'$ is non-real, $\varphi^*(Q'+\bar{Q}')$ is non-real. We have $\delta(D_1) = \delta(D) = s-2$ and D_1 is clearly a special divisor. We get $\dim|D_1| = 1 = \frac{1}{2}(d-\delta(D))-2 > \frac{1}{2}(d_1-\delta(D_1))$, hence $|D_1|$ is a very special pencil, impossible since $\delta(D_1) \neq s$.

We have $t = 2$ and thus $d' = \frac{s}{2}+2$. Let $C_1, \ldots, C_{s-2}$ (resp. C_{s-1}, C_s) denote the connected components of $X(\mathbb{R})$ on which the degree of the restriction of D is odd (resp. even). The image of a connected component of $X(\mathbb{R})$ is either a connected component of $X'(\mathbb{R})$ or a closed and bounded interval of a connected component of $X'(\mathbb{R})$. Since D is a union of fibers of $\varphi_{|D|}$ we get:
• for $i = 1, \ldots, s-2$, $\varphi(C_i)$ is a connected component of $X'(\mathbb{R})$.
• for $i = 1, \ldots, s-2$ and for $j = s-1, s$, $\varphi(C_i) \cap \varphi(C_j) = \emptyset$.
• for $i = 1, \ldots, s-2$, $\varphi^{-1}(\varphi(C_i))$ is either C_i or $C_i \cup C_{i'}$ for $i' \in \{1, \ldots, s-2\}$ distinct from i.
Let s' denote the number of connected components of $X'(\mathbb{R})$. From above remarks, we get

$$s' \geq \frac{s-2}{2} + 1 = \frac{s}{2}$$

and

$$\delta(D') \geq \frac{s-2}{2}.$$

Assume D' is special. We have

$$\dim|D'| = 3 = \dim|D| = \frac{1}{2}(d-\delta(D)) = d' - \frac{s-2}{2} \geq d' - \delta(D') \geq \frac{1}{2}(d'-\delta(D')).$$

Since $\dim|D'|$ is odd, it follows from the above inequalities that D' is very special. By Proposition 3.1, $d' = s'+4$ and $\delta(D') = s'$ and thus $\dim|D'| = 3 = d' - \frac{s-2}{2} \geq d' - \delta(D') \geq 4$, a contradiction.
Since D is non-special, Riemann-Roch gives $dim|D'| = 3 = d' - g' = \frac{s}{2} + 2 - g'$ i.e.

$$g' = \frac{s}{2} - 1.$$

Since $s' \geq \frac{s}{2}$, we get $s' = g'+1 = \frac{s}{2}$ by Harnack inequality i.e. X' is an M-curve. Moreover, from above remarks, it follows that there exist g' connected components of $X'(\mathbb{R})$ such that the inverse image by φ of each of these components is a union of two connected components of $X(\mathbb{R})$ among $C_1, \ldots, C_{s-2}$; the connected component of $X'(\mathbb{R})$ that remains contains the image of C_{s-1} and C_s.

We know that $4 \leq s \leq g-1$ by Theorem 1.2. If the g^1_4 is not induced by φ then $4 \geq g - 2g' + 1$ ([CKM, Cor. 2.2.2]) and we find again $s = g-1$. Now assume $g^1_4 = \varphi^*(h^1_2)$ for a h^1_2 on X'. We must consider two cases.
• $g' = 1$: We have $s' = 2$ and $s = 4$. The very special pencil g^1_s is clearly not induced by φ since $\delta(g^1_s) = 4$). By [CKM, Cor. 2.2.2], we obtain $s = g-1$.
• $g' > 1$: It follows in that case that X' is an hyperelliptic M-curve. Therefore there exists $P' \in X'(\mathbb{R})$ such that $h^1_2 = |2P'|$ but then $\delta(g^1_4) = 0$, a contradiction. □

Theorem 6.15 *(Very special linear systems on a 4-gonal curve with $\delta(g^1_4) = 2$) Let X be a real curve with a fixed g^1_4 with $\delta(g^1_4) = 2$ and such that $X_{\mathbb{C}}$ is 4-gonal. Let $|D|$ be a very special linear system on X then $|D|$ is a pencil and $s = g-1$.*

Proof : According to Proposition 6.14, it is sufficient to show that X must have a very special pencil in the case X has a very special linear system.

We note that a very special linear system on X cannot be of type 1 and cannot be of type 2.

Let $|D|$ be a base point free very special linear system of degree d such that $\mathrm{ind}(D) = k \geq 1$. We have

$$\dim|D| = r = \frac{1}{2}(d - \delta(D)) + k + 1.$$

By Lemma 6.13, $r \geq 4$. From Proposition 6.7, it follows that

$$\dim|D - g^1_4| = r - 2$$

and $|D - g^1_4|$ is base point free. Since $\delta(D - g^1_4) - 2 \leq \delta(D) \leq \delta(D - g^1_4) + 2$, we get

$$\dim|D - g^1_4| = \frac{1}{2}((d-4) - \delta(D)) + k + 1 \geq \frac{1}{2}(\deg(D - g^1_4) - \delta(D - g^1_4)) + (k-1) + 1$$

and it follows that $|D - g^1_4|$ is also base point free and very special. Since $r \geq 4$, $|D - g^1_4|$ is not a pencil and, according to Proposition 6.7, we obtain

$$\dim|D - 2g^1_4| = \dim|D| - 4$$

and $|D - 2g^1_4|$ is base point free. It is easy to see that $|D - 2g^1_4|$ is very special of index $k = \mathrm{ind}(D)$. If $\dim|D - 2g^1_4| \geq 4$ then repeating the same process we obtain finally a base point free very special linear system of dimension ≤ 3 and index $k \geq 1$, impossible by the Theorems 1.6, 2.4 and Proposition 3.1. Since the index of the base point free part is greater or equal than the index of a very special linear system (Lemma 2.2), it follows that the index of any very special linear system is null.

Let $|D|$ be a base point free very special linear system of degree d such that $|D|$ is not a pencil. We have

$$\dim|D| = r = \frac{1}{2}(d - \delta(D)) + 1.$$

By Lemma 6.13, $r \geq 4$ and it follows from Proposition 6.7 that

$$\dim|D - g^1_4| = r - 2$$

and $|D - g^1_4|$ is base point free. We can compare $\delta(D)$ and $\delta(D - g^1_4)$, we have 3 possibilities.

- Case $\delta(D-g_4^1)=\delta(D)+2$: We have

$$\dim|D-g_4^1|=\frac{1}{2}(deg(D-g_4^1)-\delta(D-g_4^1))+2$$

and $|D-g_4^1|$ is very special of index 1, impossible by an above conclusion.

- Case $\delta(D-g_4^1)=\delta(D)$: We have

$$\dim|D-g_4^1|=\frac{1}{2}(deg(D-g_4^1)-\delta(D-g_4^1))+1$$

and then $|D-g_4^1|$ is a base point free very special linear system. Remark that $\delta(D)=\delta(D-g_4^1)<s$ since $\delta(g_4^1)=2$. If $|D-g_4^1|$ is not a pencil the we may repeat the same process since $\delta(D-2g_4^1)=\delta(D)=\delta(D-g_4^1)$, and we finally get a very special pencil (we use Theorem 2.4 and Lemma 6.13 to exclude the case the linear system we obtain has dimension 2 or 3) with δ invariant $<s$, impossible by Theorem 1.6.

We have proved that

$$\delta(D-g_4^1)=\delta(D)-2$$

and we recall that $|D-g_4^1|$ is base point free by Proposition 6.7. We also remark that $\dim|D-g_4^1|\geq 2$. Since $|D-g_4^1|$ is base point free we have

$$r-6\leq\dim|D-2g_4^1|\leq r-3.$$

Suppose $\dim|D-2g_4^1|=\dim|D|-6\geq 0$. Then

$$\dim|K-(D-g_4^1)+g_4^1|=\dim|K-(D-2g_4^1)=\dim|K-(D-g_4^1)|$$

by Riemann-Roch, impossible since $h^0(K-(D-g_4^1))=h^1(D-g_4^1)>0$. Thus $\dim|K-(D-g_4^1)+g_4^1|\geq\dim|K-(D-g_4^1)|+1$.

Suppose $\dim|D-2g_4^1|=\dim|D|-3\geq 0$. Then

$$\dim|(D-2g_4^1)+g_4^1|=\dim|D-2g_4^1|+\dim|g_4^1|.$$

By Lemma 6.13, we get $\dim|D-2g_4^1|>0$. Let $|E|$ (resp. F) denote the base point free part (resp. the base part) of $|D-2g_4^1|$ then

$$\dim|E|+\dim|g_4^1|=\dim|D-2g_4^1|+\dim|g_4^1|=\dim|(D-2g_4^1)+g_4^1|\geq\dim|E+g_4^1|\geq\dim|E|+\dim|g_4^1|$$

i.e.

$$\dim|E+g_4^1|=\dim|E|+\dim|g_4^1|.$$

By Lemma 6.1, $|E|=(r-3).g_4^1$. It follows that $|D|=|(r-1).g_4^1+F|$ i.e. $|D|$ is a very special linear system of type 3; Proposition 6.11 gives a contradiction since $r\geq 4$.

Suppose $\dim|D-2g_4^1|=\dim|D|-5\geq 0$. By Riemann-Roch

$$\dim|K-(D-g_4^1)+g_4^1|=\dim|K-(D-g_4^1)|+\dim|g_4^1|.$$

Assume $\dim|K-(D-g_4^1)|>0$ and denote by $|E'|$ (resp. F') the base point free part (resp. the base part) of $|K-(D-g_4^1)|$. We have

$$\dim|E'|+\dim|g_4^1|=\dim|K-(D-g_4^1)|+\dim|g_4^1|$$

$$= \dim|K-(D-g_4^1)+g_4^1| \geq \dim|E'+g_4^1| \geq \dim|E'|+\dim|g_4^1|$$

i.e.

$$\dim|E'+g_4^1| = \dim|E'|+\dim|g_4^1|.$$

By Riemann-Roch, $\dim|E'| = \dim|K-(D-g_4^1)| = r-2-(d-4)+g-1 = r-d+g+1$. By Lemma 6.1, $|E'| = |(r-d+g+1).g_4^1|$. Hence $|K-(D-g_4^1)| = |K-D+g_4^1| = |(r-d+g+1).g_4^1|+F'$ and then $|K-D| = |(r-d+g).g_4^1+F'|$. It follows that $\dim|K-D| \geq r-d+g$, impossible by Riemann-Roch. Hence $\dim|K-(D-g_4^1)| = 0$ i.e. $\dim|K-D+g_4^1| = 0$, again impossible since D is special.

According to above, we have

$$\dim|D-2g_4^1| = \dim|D|-4.$$

Since $r \geq 4$ then $h^0(D-2g_4^1) > 0$ and $D-2g_4^1$ is special. Since $\delta(D) = \delta(D-2g_4^1)$ it is easy to see that $|D-2g_4^1|$ is very special. Let $|D_1|$ denote the base point free part of $|D-2g_4^1|$. By Lemma 2.2 and an above remark, $|D_1|$ is very special of index null. If $\dim|D-2g_4^1| \leq 3$ then $|D_1|$ must be a very special pencil g_s^1 and $|D-2g_4^1| = |D_1|$ since a very special pencil is primitive. If $\dim|D-2g_4^1| \geq 4$, we do the same process with $|D_1|$ replacing $|D|$. We have proved the existence of a very special pencil on X, the rest of the proof follows now from Proposition 6.14. □

In the following we describe all the very special linear sytems on a real curve with four real connected components (see Theorem 4.1).

Theorem 6.16 *(Very special linear systems on a 4-gonal curve with $\delta(g_4^1) = 4$)*
Let X be a real curve with a fixed g_4^1 with $\delta(g_4^1) = 4$ and such that $X_{\mathbb{C}}$ is 4-gonal. Let $|D|$ be a very special linear system of degree d and dimension r on X. Then $|D|$ is primitive, r is odd, $\mathrm{ind}(D) = 0$, $\delta(D) = s = 4$. Moreover, we are in one of the following cases:
• *$|D|$ is simple and $d \leq g-1$: then $r = 3$, $|D| = |g_4^1+h_4^1|$ with h_4^1 another pencil such that $\delta(h_4^1) = 0$, $\varphi_{|D|}(X)$ is a smooth curve of bidegree $(4,4)$ on a quadric surface Q of $\mathbb{P}^3$.*
• *$|D|$ is simple and $d \geq g$: then $|D| = |K-h_4^1|$ with h_4^1 a very special pencil.*
• *$|D|$ is a very special pencil h_4^1.*
• *$|D|$ is non simple and is not a pencil: then X is a bi-elliptic curve and*

$$|D| = |\frac{r-1}{2}g_4^1+h_4^1|$$

with h_4^1 a pencil such that $\delta(h_4^1) = 4$ (i.e. very special) if $r = 1 \mod 4$ and $\delta(h_4^1) = 0$ if $r = 3 \mod 4$.

Proof : Let $|D|$ be a very special linear system of degree d and dimension r. By Theorem 4.6, $|D|$ is primitive, $\mathrm{ind}(D) = 0$ and $\delta(D) = s = 4$. We have

$$\dim|D| = r = \frac{1}{2}(d-4)+1 = \frac{d}{2}-1.$$

Assume $|D|$ is simple and $d \leq g-1$. By Castelnuovo's bound

$$r \leq \frac{1}{3}(d+1)$$

and we get $d \leq 8$ and $r \leq 3$. By Theorem 2.4 and since $|D|$ is simple we get $r = 3$ and $d = 8$. By Proposition 6.7, $|D - g_4^1|$ is a base point free pencil h_4^1 with $\delta(h_4^1) = 0$. It follows that $\varphi_{|D|}(X)$ is a curve of bidegree $(4, 4)$ on a quadric surface Q of $\mathbb{P}^3$. By Proposition 3.2, $\varphi_{|D|}(X)$ is smooth i.e. $|D|$ is very ample. By [1, p. 118], the quadric containing $\varphi_{|D|}(X)$ is unique.

Assume $|D|$ is simple and $d \geq g$. By Proposition 4.2, Lemma 2.1 and Theorem 2.4, $|K - D|$ is a very special pencil h_4^1 i.e. $|D|$ is of type 2 for that h_4^1.

Assume $|D|$ is non simple and is not a pencil. By Theorem 1.6 r is odd, $\varphi_{|D|} : X \to X'$ has degree two and X' is an elliptic curve with two real connected components and the inverse image of any connected component of $X'(\mathbb{R})$ is a disjoint union of two connected components of $X(\mathbb{R})$. From Theorem [Mo2, Thm. 4.1] and using Proposition 6.7 and [CM, Example 1.13], we see that

$$|D| = |\frac{r-1}{2} g_4^1 + h_4^1|$$

with h_4^1 a pencil such that $\delta(h_4^1) = 4$ (i.e. very special) if $r = 1 \mod 4$ and $\delta(h_4^1) = 0$ if $r = 3 \mod 4$. □

From [Mo1, Prop. 2.10], Theorems 5.1, 6.9, 6.15 and 6.16, we get Theorem 1.8 stated in the introduction.

Theorem 6.17 *Let X be real curve such that X and $X_{\mathbb{C}}$ are both n-gonal with $2 \leq n \leq 4$. If X has a very special linear system then X has a very special pencil and X is a separating curve i.e. $a(X) = 0$. Moreover, if $|D|$ is a very special linear system then* $\mathrm{ind}(D) = 0$, *$\delta(D) = s$ and $|D|$ is primitive.*

7 Clifford type inequality for very special linear systems

Using the results of the previous sections, we will improve the inequalities of Theorem 1.2 and Theorem 1.4.

Theorem 7.1 *Let X be a real curve such that X is not hyperelliptic and X is not trigonal, i.e. the real gonality of X is ≥ 4. Let D be a very special divisor of degree d and index k then*

$$\dim|D| \leq \frac{d}{2} - \frac{s}{4}.$$

Proof : Let $|D|$ be a very special linear system of degree d, index k and dimension r. Before proving the inequality stated in the Theorem, we will prove the following inequality

$$\dim|D| \leq \frac{1}{2}(d - \beta(D)) - k - 1.$$

Assume $|D|$ is a pencil. By Proposition 2.5, we have $\delta(D) = s$, $d = s$ and $k = 0$. According to Propositions 4.3 and 4.4, we have $s \geq 4$ and thus

$$\dim|D| = 1 \leq \frac{1}{2}(d - \beta(D)) - k - 1 = \frac{1}{2}s - 1.$$

In the following of the proof, we assume $|D|$ is not a pencil. By Lemma [Mo1, Lem. 2.5] and Lemma 2.1, we may assume $|D|$ is base point free and $d \leq g - 1$.

We have

$$r = \frac{1}{2}(d - \delta(D)) + k + 1 \tag{1}$$

and suppose

$$r > \frac{1}{2}(d - \beta(D)) - k - 1 \tag{2}$$

By Proposition 2.9, we get

$$d + \delta(D) \geq 2s + 2k + 4 \tag{3}$$

From (2), (3) and since $\beta(D) = s - \delta(D)$, we get

$$r \geq \frac{s}{2} + \frac{3}{2} \tag{4}$$

By (1) and (2), we obtain

$$2r \geq d - \frac{s}{2} + \frac{1}{2} \tag{5}$$

Using (4) and (5), it follows that

$$3r \geq d + 2 \tag{6}$$

If $|D|$ is simple, there is a contradiction with Castelnuovo's bound $3r \leq d + 1$.

Therefore $|D|$ is non simple and we know that $\delta(D) = s$ and $k = 0$ in that case by Theorem 1.6. From By (1) and (2), we get

$$\frac{1}{2}(d - s) + 1 > \frac{1}{2}d - 1$$

i.e.

$$s \leq 3.$$

The case $s = 1$ is not possible by Theorem 1.1. The case $s = 2$ (resp $s = 3$) is impossible by Proposition 4.3 (resp. 4.4) and given the hypotheses.

Set $A = \frac{1}{2}(d - \beta(D)) - k - 1$ and $B = \frac{d}{2} - \frac{s}{4}$. Then

$$r + A = 2B.$$

Therefore, since we have proved that $r \leq A$ then

$$r \leq B \leq A$$

and the proof is done. □

We are interested by the case when we have an equality in the inequality given in the previous theorem.

We introduce a new invariant of very special linear systems.

Definition 7.2 *Let D be a very special divisor. The rational number $l \in \mathbb{Q}$ with $2l \in \mathbb{Z}$ such that*

$$\dim|D| = \frac{1}{2}(\deg(D) - \beta(D)) - l$$

is called the coindex of D (or $|D|$) and is denoted by $\mathrm{coind}(D)$.

Lemma 7.3 *[Mo1, Lem. 3.6] Let D be a very special divisor then*

$$\mathrm{coind}(D) = \mathrm{coind}(K - D).$$

We reformulate Theorems 1.2 and 1.5 using the notion of coindex.

Theorem 7.4 *[Mo1, Thm. 3.8, Thm. 3.18] Let D be a very special divisor then*

$$\mathrm{coind}(D) \geq \mathrm{ind}(D).$$

If there is an equality in the previous inequality then X is hyperelliptic with a very special g^1_2 and $|D| = r.g^1_2$ with $r = \dim|D|$ odd.

We give a consequence of the proof of Theorem 7.1.

Corollary 7.5 *Let X be a real curve such that X is not hyperelliptic and X is not trigonal, i.e. the real gonality of X is ≥ 4. Let D be a very special divisor then*

$$\mathrm{coind}(D) \geq \mathrm{ind}(D) + 1.$$

Lemma 7.6 *Let D be an effective divisor. Let F be the base divisor of $|D|$. If D is very special then the base point free part $|E| = |D - F|$ of $|D|$ is also very special and*

$$\mathrm{coind}(E) \leq \mathrm{coind}(D).$$

Moreover $\mathrm{coind}(E) = \mathrm{coind}(D - F) = \mathrm{coind}(D)$ *if and only if $F = \sum_i P_i$ with the P_i some real points among the $\beta(D)$ real connected components on which the degree of the restriction of D is even, such that no two of them belong to the same real connected component.*

Proof : Set $d = \deg(D)$ and $l = \mathrm{coind}(D)$.

Assume a non-real point $Q + \bar{Q}$ is contained in the base divisor of $|D|$. Then $\dim|D| = \dim|D - Q - \bar{Q}| = \frac{1}{2}(d - \beta(D)) - l = \frac{1}{2}((d-2) - \beta(D)) - (l-1)$ and $\mathrm{coind}(D - Q - \bar{Q}) = \mathrm{coind}(D) - 1$.

Assume two real points P, P' belonging to the same real connected component, are contained in the base divisor of $|D|$, then as before, $\mathrm{coind}(D - P - P') = \mathrm{coind}(D) - 1$.

Assume a real point P belonging to a connected component on which the degree of the restriction of D is even, is a base point of $|D|$. Then $\dim|D| = \dim|D - P| = \frac{1}{2}(d - \beta(D)) - l = \frac{1}{2}((d-1) - (\beta(D) - 1)) - l = \frac{1}{2}(\deg(D - P) - \beta(D - P)) - l$ and $\mathrm{coind}(D - P) = \mathrm{coind}(D)$.

Assume a real point P belonging to a connected component on which the degree of the restriction of D is odd, is a base point of $|D|$. Then $\dim|D| = \dim|D - P| = \frac{1}{2}(d - \beta(D)) - l = \frac{1}{2}((d-1) - (\beta(D) + 1)) - l + 1 = \frac{1}{2}(\deg(D - P) - \beta(D - P)) - (l - 1)$ and $\mathrm{coind}(D - P) = \mathrm{coind}(D) - 1$. □

Lemma 7.7 *([8, Lem. 3.1]) Let D and E be divisors of degree d and e on a curve X of genus g and suppose that $|E|$ is base point free. Then*

$$h^0(D) - h^0(D - E) \leq \frac{e}{2}$$

if $2D - E$ is special.

The previous lemma applies in case D is semi-canonical i.e. $2D = K$.

Lemma 7.8 *([Ac], [7] p. 200 and [1] p. 122) Let X be an extremal curve (it means the genus of the curve is maximal i.e. the genus equals the Castelnuovo's bound) of degree $d > 2r$ in $\mathbb{P}^r_{\mathbb{R}}$ ($r \geq 3$). Then one of the followings holds:*

(i) *X lies on a rational normal scroll Y in $\mathbb{P}^r_{\mathbb{R}}$ (Y is real, see [1] p. 120). Write $d = m(r-1)+1+\varepsilon$ where $m = [\frac{d-1}{r-1}]$ and $\varepsilon \in \{0,1,2,\ldots,r-2\}$. The curve $X_{\mathbb{C}}$ has only finitely many base point free pencils of degree $m+1$ (in fact, only 1 for $r>3$, and 1 or 2 if $r=3$). These pencils are swept out by the rulings of $Y_{\mathbb{C}}$. Moreover $X_{\mathbb{C}}$ has no g^1_m.*

(ii) *X is the image of a smooth plane curve X' of degree $\frac{d}{2}$ under the Veronese map $\mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$.*

Proposition 7.9 *Let X be a real curve such that X is not hyperelliptic and X is not trigonal, i.e. the real gonality of X is ≥ 4. Let D be a very special divisor of degree d and index k such that*

$$\dim|D| = \frac{d}{2} - \frac{s}{4}$$

or equivalently such that

$$\mathrm{coind}(D) = \mathrm{ind}(D) + 1 = k+1.$$

Then $X_{\mathbb{C}}$ is 4-gonal and X has a very special pencil g^1_4. Moreover $s=4$, $a(X)=0$, $\delta(D)=s$, $k=0$ and $|D|$ is one of the linear systems listed in Theorem 6.16.

Proof : Looking at the proof of Theorem 7.1, we see that the equality $\dim|D| = \frac{d}{2} - \frac{s}{4}$ is equivalent to the other equality $\mathrm{coind}(D) = \mathrm{ind}(D)+1$.. Thus

$$\dim|D| = r = \frac{1}{2}(d-\delta(D)) + k + 1 = \frac{1}{2}(d-\beta(D)) - k - 1.$$

We claim $|D|$ is base point free. By the Lemmas 2.2, 7.6 and Corollary 7.5, if $|E|$ denote the base point free part of $|D|$ then we must have $\mathrm{ind}(E) = \mathrm{ind}(D)$ and $\mathrm{coind}(E) = \mathrm{coind}(D)$ since $\mathrm{coind}(E) \geq \mathrm{ind}(E)+1$. It follows now from the lemmas 2.2 and 7.6 that $|E| = |D|$ since they have the same index and coindex. By Lemma 7.3, $|D|$ is primitive. By the Lemmas 2.1 and 7.3, we may assume $d \leq g-1$.

We assume first that $|D|$ is non-simple. By Theorem 1.6, $k=0$ and $\delta(D)=s$. We get

$$\frac{1}{2}(d-s)+1 = \frac{d}{2} - \frac{s}{4}$$

i.e.

$$s = 4.$$

By Theorem 4.6, X has a very special g^1_4 and $X_{\mathbb{C}}$ is 4-gonal. We use Theorem 6.16 to finish the proof in this case.

We assume now that $|D|$ is simple. We have

$$r = \frac{1}{2}(d-\delta(D)) + k + 1 \tag{7}$$

$$r = \frac{1}{2}(d-\beta(D)) - k - 1 \tag{8}$$

and

$$r = \frac{d}{2} - \frac{s}{4} \tag{9}$$

By Proposition 2.9, we get $d + \delta(D) \geq 2s + 2k + 4$ and we claim that here it is an equality:
If $d + \delta(D) \geq 2s + 2k + 6$ then using (8) we get

$$r \geq \frac{s}{2} + 2.$$

Using now (9) we have $s = 2d - 4r$ and replacing in the previous inequality we get

$$3r \geq d + 2$$

and this contradicts Castelnuovo's bound. Therefore, we have

$$d + \delta(D) = 2s + 2k + 4 \tag{10}$$

From (9), (10) and (8), it follows that

$$r = \frac{s}{2} + 1$$

and that

$$3r = d + 1$$

i.e. $\varphi_{|D|}(X)$ is an extremal curve in the sense of Castelnuovo. By [Ac2, Lem. 2.9], D is semi-canonical i.e. $|2D| = |K|$. We denote by Y the curve $\varphi_{|D|}(X)$. We have $m = [\frac{d-1}{r-1}] = [\frac{3r-2}{r-1}] = 3$ since $r \geq 3$. By Lemma 7.8, we have to consider the following cases.

Case 1 $r = 5$ and X is a smooth plane curve:
By Lemma 7.8, Y is the image of a smooth plane curve of degree 7 under the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The curve X has a unique very ample g^2_7 which calculate the Clifford index of $X_{\mathbb{C}}$. Since D is semi-canonical, by Lemma 7.7, the linear system $|D - g^2_7|$ of degree 7 has dimension ≥ 2. Since the Clifford index of X is 3, we have $\dim|D - g^2_7| = 2$. It follows that $|D| = 2g^2_7$ and $\delta(D) = 0$, impossible.

Case 2 $r \geq 4$ and X is not a smooth plane curve:
By Lemma 7.8, X has a g^1_4 and $X_{\mathbb{C}}$ is 4-gonal. From the Theorem 6.9, 6.15 and 6.16, it follows that $k = 0$ and $\delta(D) = s$. By (10) and (7), $d = s + 4$ and $r = 3$, impossible.

Case 3 $r = 3$:
By Proposition 3.1, $d = s+4$, $k = 0$ and $\delta(D) = s$. Since in this case the rational scroll is a quadric surface, the existence of the very special g^1_4 follows from Proposition 3.2. □

We summarize the results of this section in the following theorem.

Theorem 7.10 *Let $|D|$ be a very special linear system of degree d on a real curve X.*

(i) *We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s-2}{2}),$$

with equality i.e.

$$\mathrm{coind}(D) = \mathrm{ind}(D)$$

if and only if X is hyperelliptic, the g^1_2 is very special and $s = 2$.

(ii) *Assume X is not hyperelliptic. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s-1}{2}),$$

with equality i.e.

$$\mathrm{coind}(D) = \mathrm{ind}(D) + \frac{1}{2}$$

if and only if X is trigonal, a g^1_3 is very special and $s = 3$.

(iii) *Assume X is not hyperelliptic and not trigonal. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s}{2}),$$

with equality i.e.

$$\mathrm{coind}(D) = \mathrm{ind}(D) + 1$$

if and only if X is 4-gonal, a g^1_4 is very special and $s = 4$.

(iv) *Assume X has gonality ≥ 5. We have*

$$\dim|D| \leq \frac{1}{2}(d - \frac{s+1}{2}).$$

Proof: The proof of the theorem follows from the results of the paper except maybe the part concerning equality in (ii).

If X is trigonal with a very special g^1_3 then $s = 3$ and $\dim g^1_3 = \frac{1}{2}(deg(g^1_3) - \frac{s-1}{2})$.

Assume X is not hyperelliptic and suppose there is a very special linear system $|D|$ of degree d such that

$$\dim|D| = \frac{1}{2}(d - \frac{s-1}{2}).$$

By Theorems 7.1 and 1.5, X is trigonal. By Theorem 5.1, a g^1_3 is very special. □

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