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▶ **To cite this version:**

Larbi Alili, Piotr Graczyk, Tomasz Zak. On inversions and Doob h-transforms of linear diffusions. 2012. hal-00735182v2

HAL Id: hal-00735182
https://hal.science/hal-00735182v2

Preprint submitted on 10 Oct 2012

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On inversions and Doob h-transforms of linear diffusions

L. Alili[(1)], **P. Graczyk**[(2)], **T. Żak**[(3)]

ABSTRACT

Let X be a regular linear diffusion whose state space is an open interval $E \subseteq \mathbb{R}$. We consider a diffusion X^* which probability law is obtained as a Doob h-transform of the law of X, where h is a positive harmonic function for the infinitesimal generator of X on E. This is the dual of X with respect to $h(x)m(dx)$ where $m(dx)$ is the speed measure of X. Examples include the case where X^* is X conditioned to stay above some fixed level. We provide a construction of X^* as a deterministic inversion of X, time changed with some random clock. The study involves the construction of some inversions which generalize the Euclidean inversions. Brownian motion with drift and Bessel processes are considered in details.

Keywords and Phrases: Diffusion process, Dual process, Doob conditional process, Inversion, Involution.

AMS 2000 subject classification: 60G60; 60J45.

1. Introduction and preliminaries

Let $X := (X_t, t \leq \zeta)$ be a regular diffusion with life time ζ and state space $E = (l, r) \subseteq \mathbb{R}$ which is defined on complete probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Assume that X is killed, i.e. sent to a cemetery point Δ, as soon as it hits one of the boundaries; that is $\zeta = \inf\{s,\ X_s = l \text{ or } X_s = r\}$ with the usual convention $\inf\{\emptyset\} = +\infty$.

Our objectives in this paper are summarized as follows. Given a positive function h which is harmonic for the infinitesimal generator L of X, i.e. $Lh = 0$, we give an explicit construction of the dual of X with respect to $h(x)m(dx)$ where $m(dx)$ is the speed measure of X. The dual, which distribution is obtained by a Doob h-transform change of measure of the distribution of the original diffusion is either the process itself, i.e. the process is self-dual, or the original diffusion conditioned to have opposite behaviors at the boundaries

when started from a specific point x_0 in the state space; this is explained in details in Proposition 4 below. We refer to the original paper [5] by J.L. Doob for h-transforms and to [3] where this topic is surveyed. The procedure consists in first constructing the inverse of the diffusion with respect to a point $x_0 \in E$ which is a deterministic involution of the original diffusion. Time changing then with an appropriate clock gives a realization of the dual. The construction is known for the case where X is a three dimensional Bessel process started at a positive X_0. In this case, the inverse process with respect to $x_0 = 1$ is $(1/X_t)$ and the dual process is a Brownian motion killed when it hits 0, see [14]. Of course, the former process is a Brownian motion conditioned via a Doob h-transform to stay positive. Similarly, the three-dimensional hyperbolic Bessel process can be realized as a Brownian motion with negative unit drift conditioned to stay positive. In fact, this work is inspired by the discovery of the inversion $\frac{1}{2}\ln\coth X_t$ of the three dimensional hyperbolic Bessel process (X_t) with respect to $x_0 = \frac{1}{2}\ln(1+\sqrt{2})$; when this is appropriately time changed, we obtain a Brownian motion with negative unit drift. The motivation of the search for deterministic inversions between stochastic processes comes from Potential Theory, where the Kelvin inversion plays a crucial role, see e.g.[2].

In order to say more, let us fix the mathematical setting. Suppose that X satisfies the s.d.e.

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds, \quad t < \zeta, \tag{1.1}$$

where $X_0 \in E$, $(W_t, t \leq \zeta)$ is a standard Brownian motion and $\sigma, b : E \to \mathbb{R}$ are measurable real valued functions. Assume that σ and b satisfy the Engelbert-Schmidt conditions

$$\sigma \neq 0 \text{ and } 1/\sigma^2,\ b/\sigma^2 \in L^1_{loc}(E), \tag{1.2}$$

where $L^1_{loc}(E)$ is the space of locally integrable functions on E. Condition (1.2) implies that (1.1) has a unique solution in law, see Proposition 5.15 in [9]. We write $(\mathcal{F}^X_t, t \geq 0)$ for the natural filtration generated by X and denote by $\mathcal{DF}(E)$ the set of diffusions satisfying the aforementioned conditions. For background on diffusion processes, we refer to ([3], [9], [13]–[15]).

Let $X \in \mathcal{DF}(E)$. For $y \in E$, let $H_y = \inf\{t > 0;\ X_t = y\}$ be the first hitting time of y by X. Recall that the scale function of X is any continuous strictly increasing function

on E satisfying

$$\mathbb{P}_x(H_\alpha < H_\beta) = (s(x) - s(\beta))/(s(\alpha) - s(\beta)) \tag{1.3}$$

for all $l < \alpha < x < \beta < r$. This is a reference function which is strictly increasing and given modulo an affine transformation by $s(x) = \int_c^x \exp\left(-2\int_c^z b(r)/\sigma^2(r)dr\right)dz$ for some $c \in E$. For convenience, we distinguish, as in Proposition 5.22 in [9], the following four different subclasses of diffusions which exhibit different forms of inversions. We say that $X \in \mathcal{DF}(E)$ is of

- Type 1 if $-\infty < s(l)$ and $s(r) < +\infty$;
- Type 2 if $-\infty < s(l)$ and $s(r) = +\infty$;
- Type 3 if $s(l) = -\infty$ and $s(r) < +\infty$;
- Type 4 if $s(l) = -\infty$ and $s(r) = +\infty$.

Type 4 corresponds to recurrent diffusions while Types 1–3 correspond to transient ones. Recall that the infinitesimal generator of X is given by $Lf = (\sigma^2/2)f'' + bf'$ where f is in the domain $\mathcal{D}(X)$ which is appropriately defined for example in [8]. For $x_0 \in E$ let h be the unique positive harmonic function for L satisfying $h(x_0) = 1$ and either

$$h(l) = \begin{cases} 1/h(r) & \text{if X is of type 1 with } 2s(x_0) \neq s(l) + s(r); \\ 1 & \text{if X is of type 1 and } 2s(x_0) = s(l) + s(r) \text{ or type 4;} \\ 0 & \text{if X is of type 2;} \end{cases}$$

or

$$h(r) = \ 0 \quad \text{if X is of type 3.}$$

If X is of type 4 or of type 1 with $2s(x_0) = s(l)+s(r)$ then $h \equiv 1$ otherwise h is specifically given by (2.7) which is displayed in Section 3 below.

Let X^* be the dual of X, with respect to $h(x)m(dx)$, in the following classical sense. For all $t > 0$ and all Borel functions f and g, we have

$$\int_E f(x)P_t g(x)h(x)m(dx) = \int_E g(x)P_t^* f(x)h(x)m(dx)$$

where P_t and P_t^* are the semigroup operators of X and X^*, respectively. The probability law of X^* is related to that of X by a Doob h-transform, see for example [6]. To be more precise, assuming that $X_0^* = x \in E$ then the distribution $\mathbb{P}_x^*$ of X^* is obtained from the

distribution $\mathbb{P}_x$ of X by the change of measure

$$d\mathbb{P}^*_x|_{\mathcal{F}_t} = \frac{h(X_t)}{h(x)} d\mathbb{P}_x|_{\mathcal{F}_t}, \quad t < \zeta. \tag{1.4}$$

We shall denote by $\mathbb{E}^*_x$ the expectation under the probability measure $\mathbb{P}^*_x$; X^* has the infinitesimal generator $L^*f = L(hf)/h$ for $f \in \mathcal{D}(X^*) = \{g : E \to E,\ hg \in \mathcal{D}(X)\}$. The two processes $(h(X_t), t \leq \zeta)$ and $(1/h(X^*_t), t \leq \zeta^*)$ are continuous local martingales. We shall show that the former process can be realized as the latter when time changed with an appropriate random clock. Thus, the expression of either of the processes X^* and X, which are both E-valued, in terms of the other, involves the function $I(x) = h^{-1}(1/h(x)) : E \to E$ which is clearly an involution of E in the sense that $I \circ I(x) = x$, for all $x \in E$, and $I(E) = E$. For general properties of real valued involutions, we refer to [20] and [21]. Now, indeed, a variant of our main result states that when h is not constant the processes $(h(X^*_t), t \leq \zeta^*)$ and $(1/h(X_{\tau_t}), t \leq A_\zeta)$, if both are started at x_0, are equivalent, where, for $t > 0$,

$$A_t := \int_0^t I'^2(X_s)\sigma^2(X_s)/\sigma^2 \circ I(X_s)\ ds \tag{1.5}$$

and τ_t is the inverse of A_t. Two interesting features of the involved clocks are described as follows. First, $\tau_t = A^*_t$ where A^*_t is defined as above with X replaced by X^*. Second, we have ζ^* (resp. ζ) and A_ζ (resp. $A^*_{\zeta^*}$) have the same distribution; these new identities in distribution for killed diffusions resemble the Ciesielski-Taylor and Biane identities, see [1] and [4]. We call the process $(I(X_t), t < \zeta)$ the inverse of X with respect to x_0.

In the transient case, the general construction of X^* from X discovered by M. Nagasawa, in [12], applies to linear diffusions; see also [11] and [14]. We mention that this powerful method is used by M.J. Sharpe in [19] and by D. Williams in [22] for the study of path transformations of some diffusions. While the latter path transform involves time reversal from cooptional times, such as last passage time, the construction we present here involves only deterministic inversions and time changes with random clocks of the form (1.5). While we only consider one dimensional diffusions in this paper, inversions of stable processes and Brownian motion in higher dimensions are studied in [2] and [23], respectively.

This paper is organized as follows. We start section 2 with the construction of a family of inversions associated to a diffusion $X \in \mathcal{DF}(E)$. The construction involves a reference scale function s and not the speed measure of X. We note that the inversion of E in

the direction of s is uniquely characterized by the fixed point x_0. Also, among the set of inversions in the direction of s, the s-inversion with fixed point x_0 is uniquely characterized by the associated harmonic function h. Thus, the family of inversions we obtain is a one parameter family of involutions indexed by the fixed point x_0. In Section 3, we state our main result. That gives the path construction of X^* in terms of the inverse of X in the direction of s with respect to a point $x_0 \in E$. Section 4 is devoted to applications. We point out in Corollaries 2 and 3 some new results that we obtain for Bessel processes and the hyperbolic Bessel process of dimension 3.

2. Conditioned diffusions and inversions

Let $X \in \mathcal{DF}(E)$. To start with, assume that X is of type 1 and let $h : E \to \mathbb{R}_+$ be a positive harmonic function for the infinitesimal generator L of X which is not constant. Let X^* be the dual of X with respect to $h(x)m(dx)$. We are ready to state the following result which motivates the construction of inversions; to our best knowledge, the described role of x_0 and x_1 for X and X^* has not been known.

Proposition 1. *1) There exists a unique $x_0 \in E$ such that $h^2(x_0) = h(l)h(r)$. We call x_0 the h-geometric mean of $\{l, r\}$. Furthermore, for $x \in E$, we have $\mathbb{P}_x(H_l < H_r) = \mathbb{P}^*_x(H_r < H_l)$ if and only if $x = x_0$.*

2) There exists a unique $x_1 \in E$ such that $2h(x_1) = h(l) + h(r)$. We call x_1 the h-arithmetic mean of $\{l, r\}$. Furthermore, for $x \in E$, we have $\mathbb{P}_x(H_l < H_r) = \mathbb{P}_x(H_r < H_l) = 1/2$ if and only if $x = x_1$.

Proof. 1) Since h is continuous and monotone, because it is an affine function of s, if h is increasing (resp. decreasing) then the inequality $h(l) < \sqrt{h(l)h(r)} < h(r)$ (resp. $h(r) < \sqrt{h(l)h(r)} < h(l)$) implies the existence and uniqueness of x_0. Because $-1/h$ is a scale function for X^*, see for example [3], applying (1.3) yields

$$\mathbb{P}_x(H_l < H_r) = \left(h(x) - h(r)\right) / \left(h(l) - h(r)\right)$$

and

$$\mathbb{P}^*_x(H_l > H_r) = \left(1/h(x) - 1/h(l)\right) / \left(1/h(r) - 1/h(l)\right).$$

These are equal if and only if $x = x_0$. 2) The proof is omitted since it is very similar. □

To introduce the mappings we are interested in, let us express r in terms of x_0 and l, when x_0 is the h-geometric mean, to find $r = h^{-1}\left(h^2(x_0)/h(l)\right)$. This exhibits the function $I : x \to h^{-1}\left(h^2(x_0)/h(x)\right)$ which is well defined on E by monotonicity of h. Clearly, I is a decreasing involution of E with fixed point x_0. Next, observe that $h \circ I \circ h^{-1} : x \to h^2(x_0)/x$ is the Euclidian inversion with fixed point $h(x_0)$. We return now to the general case and assume that X is of either of the types 1–4. Our aim is to determine the set of all involutions associated to X which lead to the set of Möbius real involutions

$$\mathcal{MI} := \left\{\omega : \mathbb{R}\backslash\{a/c\} \to \mathbb{R}\backslash\{a/c\}; \omega(x) = (ax+b)/(cx-a), a^2+bc > 0,\ a,b,c \in \mathbb{R}\right\}.$$

Note that the condition $a^2 + bc > 0$ for $\omega \in \mathcal{MI}$ ensures that ω, when restricted to either of the intervals $(-\infty, a/c)$ and $(a/c, +\infty)$, is a decreasing involution. Let us settle the following definition.

Definition 1. *Let s and s^{-1} be, respectively, a reference scale function for X and its inverse function, and $x_0 \in E$. A mapping $I : E \to E$ is called the inversion in the direction of s, or s-inversion, with fixed point x_0 if the following hold:*

1) $I \circ I(x) = x$ *for* $x \in E$;
2) $s \circ I \circ s^{-1} \in \mathcal{MI}$;
3) $I(E) = E$;
4) $I(x_0) = x_0$.

If $s \circ I \circ s^{-1}$ is the Euclidian reflection in x_0 then I is called the s-reflection in x_0.

Since I is defined on the whole of the open interval E, it is necessarily continuous. This, in turn, implies that it is a decreasing involution such that $I(l) = r$. The objective of our next result is to show the existence of the inversion of E in the direction of s in case when s is bounded on E i.e. for diffusions of type 1.

Proposition 2. *Let $x_0 \in E$ and assume that s is bounded on E. Then, the following assertions hold.*

1) The inversion of E in the direction of s with fixed point x_0 is given by

$$I(x) = \begin{cases} s^{-1}\left(s^2(x_0)/s(x)\right) & \text{if } s^2(x_0) = s(l)s(r); \\ s^{-1}\left(2s(x_0) - s(x)\right) & \text{if } 2s(x_0) = s(l) + s(r); \\ s^{-1}\left((s(x)+a)/(bs(x)-1)\right) & \text{otherwise,} \end{cases}$$

where

$$a = \left(2s(l)s(r) - s(x_0)\left(s(l) + s(r)\right)\right)\left(s^2(x_0) - s(l)s(r)\right)^{-1} s(x_0)$$

and

$$b = \left(2s(x_0) - \left(s(l) + s(r)\right)\right)\left(s^2(x_0) - s(l)s(r)\right)^{-1}.$$

2) If $2s(x_0) \neq s(l) + s(r)$ *then* $I = h^{-1}(1/h)$ *where*

$$h(x) = \begin{cases} (bs(x) - 1)/(bs(x_0) - 1) & \text{if } s^2(x_0) \neq s(l)s(r); \\ s(x)/s(x_0) & \text{otherwise.} \end{cases}$$

Furthermore, h *is continuous, strictly monotonic and satisfies* $h \neq 0$ *on* E*.*

Proof. 1) We shall first assume that $s^2(x_0) \neq s(l)s(r)$. We look for I such that $s \circ I \circ s^{-1}(x) = (x + a)/(bx - 1)$ where a and b are reals satisfying $ab + 1 \neq 0$. Since the images of l and x_0 by I are respectively r and x_0, we get the following linear system of equations

$$\begin{cases} bs^2(x_0) - a &= 2s(x_0); \\ bs(l)s(r) - a &= s(l) + s(r). \end{cases} \tag{2.6}$$

Solving it yields a and b. We need to show that $ab + 1 \neq 0$. A manipulation of the first equation of (2.6) shows that $1 + ab = (bs(x_0) - 1)^2$. In fact, we even have the stronger fact that $s(x) \neq 1/b$ on E which is seen from $1/b > s(r)$ if $2s(x_0) > s(l) + s(r)$ and $1/b < s(l)$ if $2s(x_0) < s(l) + s(r)$. Finally, if $s^2(x_0) = s(l)s(r)$ then clearly $I(x) = s^{-1}(s^2(x_0)/s(x))$.

2) Assume that $2s(x_0) \neq s(l) + s(r)$. Let us first consider the case $s^2(x_0) \neq s(l)s(r)$. Setting $h(x) = (s(x) - 1/b)/\delta$ we then obtain

$$h^{-1}(1/h(x)) = s^{-1}\left(\left(s(x) + (b\delta^2 - 1/b)\right)/(bs(x) - 1)\right).$$

Thus, the equality $I(x) = h^{-1}(1/h)$ holds if and only if $\delta = \pm\sqrt{1 + ab}/b$ which, in turn, implies that $h(x) = \pm(bs(x) - 1)/(bs(x_0) - 1)$. Since h is positive, we take the solution with plus sign. Since h is an affine transformation of s, it is strictly monotone and continuous on E. Finally, because $s \neq 1/b$, as seen in the proof of 1), we conclude that h does not vanish on E. The case $s^2(x_0) = s(l)s(r)$ is completed by observing that this corresponds to letting $b \to \infty$ and $\delta = s(x_0)$ above which gives the desired expression for h. □

Now, we are ready to fully describe the set of inversions associated to the four types of diffusions described in the introduction. The proof of the following result is omitted, keeping in mind that when s is unbounded on E, by approximating E by a family of

intervals $(\alpha, \beta) \subset E$ where s is bounded, using continuity and letting $\alpha \to l$ and $\beta \to r$ we obtain an expression for I.

Proposition 3. *All kind of inversions of E in the direction of s with fixed point $x_0 \in E$ are described as follows.*

1) X is of type 1 with $2s(x_0) \neq s(l) + s(r)$ then the inversion is given in Proposition 2.

2) X is of type 2 then we have

$$I(x) = s^{-1}\left(s(l) + (s(x_0) - s(l))^2/(s(x) - s(l))\right).$$

3) X is of type 3 then we have

$$I(x) = s^{-1}\left(s(r) - (s(r) - s(x_0))^2/(s(r) - s(x))\right).$$

4) X is of type 4 or type 1 with $2s(x_0) = s(l) + s(r)$ then I is the s-reflection in x_0.

Going back to Proposition 1 we can express the inversions of Proposition 3 in terms of the harmonic function h instead of the reference scale function s. For that we need to compute the positive harmonic function h described in the introduction for each of the types 1–4 of diffusions. We easily get

$$h(x) = \begin{cases} \frac{bs(x)-1}{bs(x_0)-1} & \text{if } X \text{ is of type 1 and } 2s(x_0) \neq s(l) + s(r); \\ 1 & \text{if } X \text{ is of type 1 and } 2s(x_0) = s(l) + s(r) \text{ or type 4}; \\ \frac{s(x)-s(l)}{s(x_0)-s(l)} & \text{if } X \text{ is of type 2}; \\ \frac{s(r)-s(x)}{s(r)-s(x_0)} & \text{if } X \text{ is of type 3}. \end{cases} \tag{2.7}$$

Note that the case where X is of type 1 and x_0 is the s-geometric mean is covered in the first case by letting $b \to \infty$ to obtain $h(x) = s(x)/s(x_0)$. In the following result, which generalizes Proposition 1, we note that the first assertion could serve as the probabilistic definition for s-inversions.

Proposition 4. *1) A function $I : E \to E$ is the s-inversion with fixed point $x_0 \in E$ if and only if $I(E) = E$ and for all $x \in E$*

$$\mathbb{P}_{x_0}(H_x < H_{I(x)}) = \mathbb{P}^*_{x_0}(H_{I(x)} < H_x) \tag{2.8}$$

where $\mathbb{P}^*$ is the distribution of the Doob h-transform of X by some positive harmonic function h. Furthermore, The s-inversion and the h-inversion of E with fixed point x_0 are equal.

2) Let $\mathbb{Q}_{x_0}$ be the probability measure of $(I(X_t), t \leq \zeta)$ when $X_0 = x_0$. Then, clearly, formula (2.8) holds true when $\mathbb{P}_{x_0}$ is replaced by $\mathbb{Q}_{x_0}$. We call the process $(I(X_t), t \leq \zeta)$ the inverse with respect to x_0 of $(X_t, t \leq \zeta)$.

Proof. 1) If x_0 is the s-arithmetic mean of $\{l, r\}$ or X is of type 4 then we are looking for $I : E \to E$ such that $\mathbb{P}_{x_0}(H_x < H_{I(x)}) = 1/2$. Using (1.3) we get $(s(x_0) - s \circ I(x))/(s(x) - s \circ I(x)) = 1/2$ which gives that I is the s-reflection. For the other cases, using (2.8) and the fact that $-1/h$ is a scale function for X^*, we find that $I(x) = h^{-1}(h(x_0)^2/h(x))$ so that I is an h-inversion with fixed point x_0. The "only if" part is straightforward following a similar reasoning to that of the proof of Proposition 1 giving I to be either the h-reflection or the h-inversion with fixed point x_0.
2) This is easily seen by using the first assertion. □

For completeness, we show in the following Proposition how to define rigorously conditioning a transient diffusion to hit one boundary of an interval before another. For conditioning a transient diffusion to avoid one of the boundaries we refer, for example, to ([7], [10], [18]).

Proposition 5. *Assume that X is transient. Let h be as given by (2.7) and X^* be the dual of X with respect to $h(x)m(dx)$. Let p be the probability that X, when started at x_0, hits l before r. X^* is X conditioned to hit l before r with probability $q = 1 - p$.*

Proof. By construction, we have $h(x_0) = 1$. Assume at first that X is of type 1. Let us decompose h, in terms of h_l and h_r which are defined below, as follows

$$h(x) = q^* \frac{h(x) - h(l)}{h(x_0) - h(l)} + p^* \frac{h(r) - h(x)}{h(r) - h(x_0)} := q^* h_r(x) + p^* h_l(x)$$

where

$$q^* = \frac{h(x_0) - h(l)}{h(r) - h(l)} h(r) = \frac{h^*(x_0) - h^*(l)}{h^*(r) - h^*(l)} = \mathbb{P}^*_{x_0}(H_r < H_l)$$

and

$$p^* = \frac{h(r) - h(x_0)}{h(r) - h(l)} h(l) = \frac{h^*(r) - h^*(x_0)}{h^*(r) - h^*(l)} = \mathbb{P}^*_{x_0}(H_l < H_r).$$

But, we have that $q^* = p$ and $p^* = q$ when X and X^* are started at x_0. Thus, for any bounded $\mathcal{F}_t^X$-measurable functional G and $t > 0$, we can write

$$\begin{aligned}\mathbb{E}^*_{x_0}[G(X_s, s \le t), t < \zeta] &= \mathbb{E}_{x_0}[h(X_t)G(X_s, s \le t), t < \zeta] \\ &= q\mathbb{E}_{x_0}[h_l(X_t)G(X_s, s \le t), t < \zeta] \\ &+ p\mathbb{E}_{x_0}[h_r(X_t)G(X_s, s \le t), t < \zeta].\end{aligned}$$

Next, since our assumptions imply that $p = \mathbb{P}_{x_0}(H_l < H_r) \in (0,\, 1)$, we have

$$\begin{aligned}\mathbb{E}_{x_0}[h_l(X_t)G(X_s, s < t), t < \zeta] &= \mathbb{E}_{x_0}[G(X_s, s \le t)\frac{\mathbb{P}_{X_t}[H_l < H_r]}{\mathbb{P}_{x_0}(H_l < H_r)}, t < \zeta] \\ &= \mathbb{E}_{x_0}[G(X_s, s \le t), t < \zeta | H_l < H_r]\end{aligned}$$

where we used the strong Markov property for the last equality. Similarly, for the other term, since $q \in (0,\, 1)$ we get

$$\mathbb{E}_{x_0}[(h_r(X_t)G(X_s, s \le t), t < \zeta] = \mathbb{E}_{x_0}[G(X_s, s \le t), t < \zeta | H_r < H_l].$$

The last two equations imply our assertion. Assume now that $h(r) = \infty$. Then $h(l) = 0$ and $\mathbb{P}_{x_0}$-a.s. all trajectories of the process X tend to l and $p = \mathbb{P}_{x_0}(H_r < H_l) = 0$. We follow [18] to define X conditioned to avoid l as follows. For any bounded $\mathcal{F}_t^X$-measurable functional G and $t > 0$, we set

$$\begin{aligned}\mathbb{E}^*_{x_0}[G(X_s, s \le t), t < \zeta] &= \lim_{a \to r} \mathbb{E}_{x_0}[G(X_s, s \le t), t < \zeta | H_a < H_l] \\ &= \lim_{a \to r} \mathbb{E}_{x_0}[G(X_s, s \le t), t < H_a < H_l] / \mathbb{P}_{x_0}(H_a < H_l) \\ &= \lim_{a \to r} \mathbb{E}_{x_0}[h_r(X_t)G(X_s, s < t), t < H_a \wedge H_l] \\ &= \mathbb{E}_{x_0}[h(X_t)G(X_s, s \le t), t < \zeta]\end{aligned}$$

where we used the strong Markov property for the third equality and the monotone convergence theorem for the last one. In this case $\mathbb{P}^*_{x_0}$-a.s all trajectories of the process X^* tend to r and $p^* = \mathbb{P}^*_{x_0}(H_l < H_r) = 0$ which completes the proof of the statement. The case $h(l) = -\infty$ and $h(r) = 0$ can be treated similarly. □

Remark 1. *Observe that the s-inversions described in Proposition 3 solve $G(x, y) = 0$ in y, where G is the symmetric function $G(x, y) = As(x)s(y) - B(s(x) + s(y)) - C$ for some reals A, B and C. This is in agreement with the fact that I is an involution, see* [21].

Remark 2. *The inversion in the direction of s with fixed point x_0 does not depend on the particular choice we make of s. Tedious calculations show that the inversion of E in the direction of s is invariant under a Möbius transformation of s.*

Remark 3. *From the point of view of Martin boundaries, the functions h_l and h_r which appear in the proof of Proposition 5 are the minimal excessive functions attached to the boundary points l and r, see ([3], [16], [17]). That is the Doob h-transformed processes using h_l and h_r tend, respectively, to l and r. Harmonic functions having a representing measure with support not included in the boundary set of E are note considered in this paper since we do not allow killings inside E.*

3. Inversion of diffusions

Let $X \in \mathcal{DF}(E)$ and s be a scale function for X. For $x_0 \in E$, let $I : E \to E$ be the inversion of E in the direction of s with fixed point x_0. Let h be the positive harmonic function specified by (2.7). Let X^* be the dual of X with respect to $h(x)m(dx)$. As aforementioned, the distribution of X^* is obtained as a Doob h-transform of the distribution of X by using the harmonic function h, as given in (1.4). Clearly, if X is of type 1 (resp. of type 2 and drifts thus to l, of type 3 and drifts thus to r or of type 4) then X^* is of type 1 (resp. of type 3 and drifts thus to r, of type 2 and drifts thus to l or of type 4). Recall that, for a fixed $t < \zeta$, τ_t is the inverse of the strictly increasing and continuous additive functional $A_t := \int_0^t I'^2(X_s)\sigma^2(X_s)/\sigma^2 \circ I(X_s)\ ds$; τ_t^* and A_t^* are the analogue objects associated to the dual X^*. Recall that the speed measure $m(dy) = 2dy/(\sigma^2 s')$ of X is uniquely determined by

$$\mathbb{E}_x[H_\alpha \wedge H_\beta] = \int_J G_J(x,y)\, m(dy)$$

where

$$G_J(x,y) = c(s(x \wedge y) - s(\alpha))(s(\beta) - s(x \vee y))/(s(\beta) - s(\alpha))$$

for any $J = (\alpha, \beta) \subsetneq E$ and all $x, y \in J$, where c is a normalization constant and $G_J(.,.)$ is the potential kernel density relative to $m(dy)$ of X killed when it exits J; see for example [14] and [15]. Recall that $-1/h$ is a scale function and $h^2(x)m(dx)$ is the speed measure of X^*. We are ready to state the main result in this paper. The proof we give is based on the resolvent method for the identification of the speed measure, see ([14], [15], [19]). Other possible methods of proof are commented in Remarks 4 and 5.

Theorem 1. *With the previous setting, let I be the s-inversion of E with fixed point $x_0 \in E$. Assume that $X_0, X_0^* \in E$ are such that $I(X_0) = X_0^*$. Then the following assertions hold true.*

(1) For all $t < \zeta$, τ_t and A_t^* have the same distribution.

(2) The processes $(X_t^*, t \leq \zeta^*)$ and $(I(X_{\tau_t}), t \leq A_\zeta)$ are equivalent.

(3) The processes $(X_t, t \leq \zeta)$ and $(I(X^*_{\tau_t^*}), t \leq A^*_{\zeta^*})$ are equivalent.

Proof. 1) Let $t > 0$ be fixed and set $\eta_t = I(X_{\tau_t})$. Because τ_t is the inverse of A_t, we can write $A_{\tau_t} = t$. Differentiating and extracting the derivative of τ_t yields

$$\frac{d}{dt}\tau_t = \frac{\sigma^2 \circ I(X_{\tau_t})}{I'^2(X_{\tau_t})\sigma^2(X_{\tau_t})}.$$

Integrating yields

$$\tau_t = \int_0^t \left(I'\sigma/(\sigma \circ I)\right)^{-2}(X_{\tau_s})ds = A_t^\eta.$$

The proof of the first assertion is complete once we have shown that η and X^* have the same distribution which will be done in the next assertion.

2) First, assume that h is not constant. In this case, $x \to -1/h(x)$ is a scale function for η since $-1/h \circ I(X_{\tau_t}) = -h(X_{\tau_t})$ is a continuous local martingale. Next, let $J = (\alpha, \beta)$ be an arbitrary subinterval of E. We proceed by identifying the speed measure of η on J. By using the fact that the hitting time H_y^η of y by η equals $A_{H_{I(y)}}$ for $y \in E$, we can write

$$\begin{aligned}
\mathbb{E}_{I(x)}\left[H_\alpha^\eta \wedge H_\beta^\eta\right] &= \mathbb{E}_{I(x)}\left[\int_0^{H_{I(\alpha)} \wedge H_{I(\beta)}} dA_r\right] \\
&= \int_{I(\beta)}^{I(\alpha)} G_{I(J)}(I(x), y) I'^2(y)\sigma^2(y)\{\sigma^2 \circ I(y)\}^{-1} m(dy) \\
&= 2\int_{I(\beta)}^{I(\alpha)} G_{I(J)}(I(x), y) I'^2(y)\{\sigma^2 \circ I(y)s'(y)\}^{-1} dy \\
&= 2\int_\alpha^\beta G_{I(J)}(I(x), I(y))\{\sigma^2(y)(-h \circ I)'(y)\}^{-1} dy.
\end{aligned}$$

On the one hand, we readily check that $\{\sigma^2(y)(s \circ I)'(y)\}^{-1}dy = h^2(y)m(dy) = m^*(dy)$ for $y \in J$. On the other hand, we have

$$\begin{aligned}
G_{I(J)}(I(x), I(y)) &= \frac{(h(I(x) \wedge I(y)) - h(I(\beta)))(h(I(\alpha)) - h(I(x) \vee I(y)))}{h(I(\alpha)) - h(I(\beta))} \\
&= \frac{(-h^*(x \vee y) + h^*(\beta))(-h^*(\alpha) + h^*(x \wedge y))}{-h^*(\alpha) + h^*(\beta)} \\
&= G_J^*(x, y),
\end{aligned}$$

where $h^* = -1/h$ and G_J^* is the potential kernel density of X^* relative to $m^*(dy)$. The case when h is constant can be dealt with similarly but by working with s instead of h. This shows that the speed measure of η is the same as that of X^* in all cases. Now, since $\tau_t^X = A_t^\eta$ we get $\tau_t^X = A_t^*$ which, in turn, implies that $A_t^X = \tau_t^*$. Finally, using the fact that I is an involution gives

$$H_y^* = \inf\{s > 0, I(X_{A_s^*}) = y\} = \tau^*_{H_{I(y)}} = A_{H_{I(y)}}$$

for $y \in E$. The assertion is completed by letting y tend to either of the boundaries to find $\zeta^* = A_\zeta$ and $\zeta = A^*_{\zeta^*}$ as desired.

3) The proof is easy using 1) and 2), the fact that I is an involution and time changes.

□

If s is bounded at one of the boundaries and unbounded at the other, i.e. X is either of type 2 or of type 3, then formula (1.3) gives that one of the boundaries is almost surely hit before the other one. The lifetimes of X^* and X are in this case given as follows.

Corollary 1. *If X is of type 2 then $\zeta = H_l$ and A^*_∞ have the same distribution and $\zeta^* = \infty$. Similarly, if X is of type 3 then $\zeta^* = H_l^*$ and A_∞ have the same distribution and $\zeta = \infty$.*

Remark 4. *Since X satisfies the s.d.e. (1.1), by Girsanov's theorem, we see that X^* satisfies $Y_t = X_0^* + \int_0^t \sigma(Y_s)dB_s + \int_0^t (b + \sigma^2 h'/h)(Y_s)ds$ for $t < \zeta^*$ where B is a Brownian motion which is measurable with respect to the filtration generated by X^* and $\zeta^* = \inf\{s,\ Y_s = l \text{ or } Y_s = r\}$, see for example* [6]*. Long calculations show that η also satisfies the above s.d.e. which, by Engelbert-Schmidt condition (1.2), has a unique solution in law. This gives a second proof of Theorem 1. Note that the use of Itô's formula for η is licit since $I \in \mathcal{C}^2(E)$.*

Remark 5. *Another way to view the main statements of Theorem 1 is the equality of generators*

$$\frac{1}{I'^2(x)}\frac{\sigma^2(x)}{\sigma^2(I(x))}L(f \circ I)(I(x)) = \frac{1}{h(x)}L(hf)(x) = L^*f(x)$$

and

$$\frac{1}{I'^2(x)}\frac{\sigma^2(x)}{\sigma^2(I(x))}L^*(g \circ I)\,(I(x)) = Lg(x)$$

for all $x \in E$, $g \in \mathcal{D}(X)$ and $f \in \mathcal{D}(X^*)$. However, the main difficulty of this method of proof is the precise description of domains of generators.

4. Applications

4.1. **Inversions of Brownian motions killed upon exiting intervals.** Assume that X is a Brownian motion killed upon exiting the interval $E = (l, r)$. Let $X_0 = x_0 \in E$. If E is bounded then we obtain the inversions

$$I(x) = \begin{cases} x_0^2/x, & \text{if } x_0^2 = lr; \\ 2x_0 - x, & \text{if } 2x_0 = l + r; \\ (x + a)/(bx - 1), & \text{otherwise,} \end{cases}$$

where

$$a = \frac{2lr - x_0(l + r)}{x_0^2 - lr} x_0 \quad \text{and} \quad b = \frac{2x_0 - (l + r)}{x_0^2 - lr},$$

where B is a standard Brownian motion. We distinguish three cases appearing in the form of the inversion I. In the first case equation $x_0^2 = lr$ implies that l and r are of the same sign. If $l > 0$ (resp. $l < 0$) then X^* is the (resp. negative) three dimensional Bessel process killed upon exiting E. In the second case, X^* is a Brownian motion killed when it exits E. In the third case, X^* satisfies the s.d.e. $X_t^* = B_t + x_0 + \int_0^t (X_s^* - 1/b)^{-1} ds$ for $t < \zeta^*$. If x_0 is below the arithmetic mean i.e. $x_0 < (l+r)/2$ then $x_0 - 1/b = (x_0 - r)(x_0 - l)/(2x_0 - (l+r))$. By uniqueness of the solution to the s.d.e. $R_t = R_0 + B_t + \int_0^t (1/R_s) ds$ deriving the three-dimensional Bessel process R, we get that $X_t^* = 1/b + R_t$ with $R_0 = x_0 - 1/b$ killed when R exits the interval $(l - 1/b, r - 1/b)$. If x_0 is above the arithmetic mean then we find $X_t^* = 1/b - R_t$, where R is a three-dimensional Bessel process started at $1/b - x_0$, where R is killed as soon as it exits the interval $(1/b - r, 1/b - l)$.

If l is finite and $r = \infty$ or $l = -\infty$ and r is finite then by Proposition 3 we respectively obtain

$$I(x) = l + \frac{(x_0 - l)^2}{x - l} \text{ and } I(x) = r - \frac{(r - x_0)^2}{r - x}.$$

If $r = +\infty$ then a similar reasoning as above gives that $X_t^* = l + R_t$ where R is a three-dimensional Bessel process started at $x_0 - l$. If $l = -\infty$ we obtain $X_t^* = r - R_t$ where R is a three-dimensional Bessel process started at $r - x_0$. If $2x_0 = l + r$ or $l = -\infty$ and $r = +\infty$ then we obtain the Euclidian reflection in x_0, i.e. $x \to 2x_0 - x$, and X^* is a Brownian motion killed when it exits E. Note that for $E = (0, \infty)$ the conclusion from our Theorem 1 is found in Lemma 3.12, p. 257, of [14]. That is $(X_t, t \le H_0)$ is distributed as $(1/X^*_{\tau_t^*}, t \le A^*_\infty)$, where τ_t^* is the inverse of $A_t^* = \int_0^t (X_s^*)^{-4} ds$. In this case X^* is the three dimensional Bessel process; see our last Example 4.3 for Bessel processes of other dimensions. If $E = \mathbb{R}$ then I is the Euclidian reflection in x_0. Finally, observe that the set

of inversions of E we obtain for the Brownian motion killed when it exits E is precisely $\mathcal{MI}(E) := \{I \in \mathcal{MI}\,;\, I(E) = E\}$.

4.2. **Inversions of drifted Brownian motion and hyperbolic Bessel process of dimension 3.** Set $B_t^{(\mu)} = B_t + \mu t$, $t \geq 0$, where B is a standard Brownian motion and $\mu \in \mathbb{R}$, $\mu \neq 0$. Thus, $B^{(\mu)}$ is a transient diffusion which drifts to $+\infty$ (resp. to $-\infty$) if $\mu > 0$ (resp. $\mu < 0$). Let us take the reference scale function $s(x) = -e^{-2\mu x}/(2\mu)$. Observe that s is increasing for all $\mu \neq 0$. Moreover $\lim_{x\to\infty} s(x) = 0$ if $\mu > 0$ and $\lim_{x\to-\infty} s(x) = 0$ if $\mu < 0$. We take X to be $B^{(\mu)}$ killed when it exits $(l, r) \subseteq \mathbb{R}$. Let us fix $x_0 \in E$.

If we take $E = \mathbb{R}$ then by Proposition 3, even though X is of type 2 if $\mu < 0$ and of type 3 if $\mu > 0$, the inversion of E in the direction of s is the Euclidian reflection in x_0. X^* is the Brownian motion with drift $\mu^* = -\mu$ in this case. If $s(l)$ and $s(r)$ are finite then using Proposition 2 we obtain appropriate, but in most cases complicated, formulas for I. For Brownian motion with drift the case of the half-line is the most interesting. Take for instance $E = (0, \infty)$ and process $X_t = x_0 + B_t + \mu t$ starting from some point $x_0 > 0$ and killed at zero. We consider two cases: if $\mu < 0$ then $s(0) = -1/(2\mu) > 0$, $s(\infty) = \infty$ and X is of type 2; if $\mu > 0$ then $s(0) = -1/(2\mu) < 0$, $s(\infty) = 0$ and X is of type 1.

First, let $\mu < 0$. If X is of type 2 then, by Proposition 3, we have only one possible inversion: $I(x) = s^{-1}\left(s(l) + (s(x_0) - s(l))^2/(s(x) - s(l))\right)$, which gives

$$I(x) = (2|\mu|)^{-1} \ln\left((e^{-2\mu x} - 1 + (1 - e^{-2\mu x_0})^2)/(e^{-2\mu x} - 1)\right).$$

If we choose $x_0 = (2|\mu|)^{-1}\ln(1 + \sqrt{2})$ then the above formula simplifies to

$$I(x) = (2|\mu|)^{-1} \ln\left((e^{-2\mu x} + 1)/(e^{-2\mu x} - 1)\right) = (2|\mu|)^{-1} \ln \coth(|\mu| x).$$

Now, if $\mu > 0$ then X is of type 1. Because $e^{-2\mu x} \neq 0$ implies $s^2(x_0) \neq s(0)s(\infty) = 0$ only two cases are possible. Either $2s(x_0) = s(0) + s(\infty) = s(0)$, which gives $x_0 = \frac{1}{2\mu}\ln 2$, and then we have s-reflection $I(x) = -\frac{1}{2\mu}\ln(1 - e^{-2\mu x})$ or $2s(x_0) \neq s(0) + s(\infty)$ and then the formula from Proposition 3 gives the inversion

$$I(x) = \frac{1}{2\mu} \ln\left(\frac{1 + e^{-2\mu x}(e^{4\mu x_0} - 2e^{2\mu x_0})}{1 - e^{-2\mu x}}\right).$$

This in turn simplifies if we choose $e^{4\mu x_0} - 2e^{2\mu x_0} = 1$, that is, if $x_0 = \frac{1}{2\mu}\ln(1 + \sqrt{2})$. Then

$$I(x) = \frac{1}{2\mu} \ln\left(\frac{1 + e^{-2\mu x}}{1 - e^{-2\mu x}}\right) = \frac{1}{2\mu} \ln \coth(\mu x).$$

It is easy to check that if $\mu < 0$, then $h(x) = \frac{e^{-2\mu x}-1}{\sqrt{2}}$ and then X^*, being an h-process, has generator $L^*f(x) = \frac{1}{2}f''(x) + |\mu| \coth(|\mu|x)f'(x)$. In particular, if $\mu = -1$ then $I(x) = \frac{1}{2}\ln\coth x$ and X^* has generator $L^*f(x) = \frac{1}{2}f''(x) + \coth(x)f'(x)$. This recovers the well-known fact that $B_t - t$ conditioned to avoid zero is a three-dimensional hyperbolic Bessel process. The novelty here is that we get X^* as a time changed inversion of $B_t - t$. If $\mu > 0$, then $h(x) = \frac{e^{-2\mu x}+1}{\sqrt{2}}$ and X^* has the generator $L^*f(x) = \frac{1}{2}f''(x) + \mu \tanh(\mu x)f'(x)$. Note that we have shown the following particular result.

Corollary 2. *Let X be a three-dimensional hyperbolic Bessel process on $(0, \infty)$ and $I(x) = \frac{1}{2}\ln\coth x$. Then, $I(X)$ is a time-changed drifted Brownian motion $B_t - t$ conditioned to avoid zero. In particular, the functional*

$$A_\infty = \frac{1}{4}\int_0^\infty \frac{ds}{(\cosh(X_s)\sinh(X_s))^2}$$

has the same distribution as the first hitting time of 0 by the Brownian motion with minus unity drift.

4.3. **Inversions of Bessel processes.** It is known that the Bessel processes of dimensions δ and $4 - \delta$ are dual one to another, see e.g. ([3], [6], [14]). But, except the case $\delta = 1$, it was not known that, up to a time change, they can be obtained one from another by the deterministic inversion $x \to 1/x$.

Let X be a Bessel process of dimension $\delta \geq 2$. Thus, $E = (0, \infty)$, 0 is polar and X has the infinitesimal generator $Lf(x) = \frac{1}{2}f''(x) + (\delta - 1)/(2x)f'(x)$ for $x \in E$, see ([14], [6]). Let $\nu = (\delta/2) - 1$ be the index of X. The scale function of X may then be chosen to be

$$s(x) = \begin{cases} -x^{-2\nu} & \text{if } \nu > 0; \\ 2\log x & \text{if } \nu = 0; \\ x^{-2\nu} & \text{otherwise.} \end{cases}$$

X is recurrent if and only if $\delta = 2$, see for example Proposition 5.22, p. 355 in [9]. First, when $\delta = 2$ the inversion of E in the direction of s is the s-reflection $x \to x_0^2/x$. For the other cases, the inversion of E in the direction of s with fixed point 1 is found to be $x \to 1/x$. Furthermore, X^* is a Bessel process of dimension $4 - \delta < 2$. Observe that in the two considered cases the involved clock is $A_t = \int_0^t (x_0/X_s)^4\, ds$. Hence, we have shown the following result.

Corollary 3. *Let X be a Bessel process of dimension $\delta \in \mathbb{R}$, killed at 0 if $\delta < 2$, starting from $X_0 > 0$ and $I(x) = 1/x$. Then $I(X)$ is a time-changed Bessel process of dimension $4 - \delta$. In particular, the functional $\int_0^\infty ds/X_s^4$, when X is a Bessel process of dimension $\delta > 2$, has the same distribution as the first hitting time of zero by the Bessel process of dimension $4 - \delta$.*

Acknowledgment: We are greatly indebted to l'Agence Nationale de la Recherche for the research grant ANR-09-Blan-0084-01.

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