



HAL
open science

On the genealogy of conditioned stable Lévy forests

Loic Chaumont, Juan Carlos Pardo Millan

► **To cite this version:**

Loic Chaumont, Juan Carlos Pardo Millan. On the genealogy of conditioned stable Lévy forests. 2007.
hal-00155592

HAL Id: hal-00155592

<https://hal.science/hal-00155592>

Preprint submitted on 18 Jun 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the genealogy of conditioned stable Lévy forests.

June 18, 2007

L. Chaumont¹ and J.C. Pardo²

Abstract

We give a realization of the stable Lévy forest of a given size conditioned by its mass from the path of the unconditioned forest. Then, we prove an invariance principle for this conditioned forest by considering k independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n . We prove that when n and k tend towards $+\infty$, under suitable rescaling, the associated coding random walk, the contour and height processes converge in law on the Skorokhod space respectively towards the “first passage bridge” of a stable Lévy process with no negative jumps and its height process.

KEY WORDS AND PHRASES: Random tree, conditioned Galton-Watson forest, height process, coding random walk, stable Lévy process, weak convergence.

MSC 2000 subject classifications: 60F17, 05G05, 60G52, 60G17.

¹LAREMA, Département de Mathématiques, Université d'Angers, 2, Bd Lavoisier - 49045, ANGERS CEDEX 01. E-mail: loic.chaumont@univ-angers.fr

²Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4, Place Jussieu - 75252 PARIS CEDEX 05. E-mail: pardomil@ccr.jussieu.fr

1 Introduction

The purpose of this work is to study some remarkable properties of stable Lévy forests of a given size conditioned by their mass.

A Galton-Watson tree is the underlying family tree of a given Galton-Watson process with offspring distribution μ started with one ancestor. It is well-known that if μ is critical or subcritical, the Galton-Watson process is almost surely finite and therefore, so is the corresponding Galton-Watson tree. In this case, Galton-Watson trees can be coded by two different discrete real valued processes: the height process and the contour process whose definition is recalled here in section 2. Both processes describe the genealogical structure of the associated Galton-Watson process. They are not Markovian but can be written as functionals of a certain left-continuous random walk whose jump distribution depends on the offspring distribution μ . In a natural way, Galton-Watson forests are a finite or infinite collections of independent Galton-Watson trees.

The definition of Lévy trees bears upon on the continuous analogue of the height process of Galton-Watson trees introduced by Le Gall and Le Jan in [21] as a functional of a Lévy process with no negative jumps. Our presentation owes a lot to the recent paper of Duquesne and Le Gall [11], which uses the formalism of \mathbb{R} -trees to define Lévy trees that were implicit in [8], [9] and [21]. We may consider Lévy trees as random variables taking values in the space of compact rooted \mathbb{R} -trees. In a recent paper of Evans, Pitman and Winter [13], \mathbb{R} -trees are studied from the point of view of measure theory. Informally an \mathbb{R} -tree is a metric space (\mathcal{T}, d) such that for any two points σ and σ' in \mathcal{T} there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. In [13], the authors also established that the space \mathbb{T} of equivalent classes of (rooted) compact real trees, endowed with the Gromov-Hausdorff metric, is a Polish space. This makes it very natural to consider random variables or even random processes taking values in the space \mathbb{T} . In this work, we define Lévy forests as Poisson point processes with values in the set of \mathbb{R} -trees whose characteristic measure is the law of the generic Lévy tree.

First, we are interested in the construction of Lévy forests of a given size conditioned by their mass. Again, in the discrete setting this conditioning is easier to define; the conditioned Galton-Watson forest of size k and mass n is a collection of k independent Galton-Watson trees with total progeny equal to n . In section 4, we provide a definition of these notions for Lévy forest. Then, in the stable case, we give a construction of the conditioned stable Lévy forest of size $s > 0$ and mass 1 by rescaling the unconditioned forest of a particular random mass.

In [1], Aldous showed that the Brownian random tree (or continuum random tree) is the limit as n increases of a rescaled critical Galton-Watson tree conditioned to have n vertices whose offspring distribution has a finite variance. In particular, Aldous proved that the discrete height process converges on the Skorokhod space of càdlàg paths to

the normalized Brownian excursion. Recently, Duquesne [8] extended such results to Galton-Watson trees whose offspring distribution is in the domain of attraction of a stable law with index α in $(1, 2]$. Then, Duquesne showed that the discrete height process of the Galton Watson tree conditioned to have a deterministic progeny, converges as this progeny tends to infinity on the Skorokhod space to the normalized excursion of the height process associated with the stable Lévy process.

The other main purpose of our work is to study this convergence in the case of a finite number of independent Galton-Watson trees, this number being an increasing function of the progeny. More specifically, in Section 5, we establish an invariance principle for the conditioned forest by considering k independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n . When n and k tend towards ∞ , under suitable rescaling, the associated coding random walk, the contour and height processes converge in law on the space of Skorokhod towards the first passage bridge of a stable Lévy process with no negative jumps and its height process.

In section 2, we introduce conditioned Galton-Watson forests and their related coding first passage bridge, height process and contour process. Section 3 is devoted to recall the definitions of real trees and Lévy trees and to state a number of important results related to these notions.

2 Discrete trees and forests.

In all the sequel, an element u of $(\mathbb{N}^*)^n$ is written as $u = (u_1, \dots, u_n)$ and we set $|u| = n$. Let

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where $\mathbb{N}^* = \{1, 2, \dots\}$ and by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The concatenation of two elements of \mathbb{U} , let us say $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ is denoted by $uv = (u_1, \dots, u_n, v_1, \dots, v_m)$. A discrete rooted tree is an element τ of the set \mathbb{U} which satisfies:

- (i) $\emptyset \in \tau$,
- (ii) If $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$.
- (iii) For every $u \in \tau$, there exists a number $k_u(\tau) \geq 0$, such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

In this definition, $k_u(\tau)$ represents the number of children of the vertex u . We denote by \mathbb{T} the set of all rooted trees. The total cardinality of an element $\tau \in \mathbb{T}$ will be denoted by $\zeta(\tau)$, (we emphasize that the root is counted in $\zeta(\tau)$). If $\tau \in \mathbb{T}$ and $u \in \tau$, then we define the shifted tree at the vertex u by

$$\theta_u(\tau) = \{v \in \mathbb{U} : uv \in \tau\}.$$

We say that $u \in \tau$ is a leaf of τ if $k_u(\tau) = 0$.

Then we consider a probability measure μ on \mathbb{Z}_+ , such that

$$\sum_{k=0}^{\infty} k\mu(k) \leq 1 \quad \text{and} \quad \mu(1) < 1.$$

The law of the Galton-Watson tree with offspring distribution μ is the unique probability measure \mathbb{Q}_μ on \mathbb{T} such that:

(i) $\mathbb{Q}_\mu(k_\emptyset = j) = \mu(j)$, $j \in \mathbb{Z}_+$.

(ii) For every $j \geq 1$, with $\mu(j) > 0$, the shifted trees $\theta_1(\tau), \dots, \theta_j(\tau)$ are independent under the conditional distribution $\mathbb{Q}_\mu(\cdot | k_\emptyset = j)$ and their conditional law is \mathbb{Q}_μ .

A Galton-Watson forest with offspring distribution μ is a finite or infinite sequence of independent Galton-Watson trees with offspring distribution μ . It will be denoted by $\mathcal{F} = (\tau_k)$. With a misuse of notation, we will denote by \mathbb{Q}_μ the law on $(\mathbb{T})^{\mathbb{N}^*}$ of a Galton-Watson forest with offspring distribution μ .

It is known that the G-W process associated to a G-W tree or forest does not code entirely its genealogy. In the aim of doing so, other (coding) real valued processes have been defined. Amongst such processes one can cite the *contour process*, the *height process* and the associated random walk which will be called here *the coding walk* and which is sometimes referred to as the Luckazievicks path.

Definition 1. We denote by $u_\tau(0) = \emptyset$, $u_\tau(1) = 1, \dots, u_\tau(\zeta - 1)$ the elements of a tree τ which are enumerated in the lexicographical order (when no confusion is possible, we will simply write $u(n)$ for $u_\tau(n)$). Let us denote by $|u(n)|$ the rank of the generation of a vertex $u(n) \in \tau$.

(1) The height function of a tree τ is defined by

$$n \mapsto H_n(\tau) = |u(n)|, \quad 0 \leq n \leq \zeta(\tau) - 1.$$

(2) The height function of a forest $\mathcal{F} = (\tau_k)$ is defined by

$$\begin{aligned} n \mapsto H_n(\mathcal{F}) &= H_{n - (\zeta(\tau_0) + \dots + \zeta(\tau_{k-1}))}(\tau_k), \\ &\text{if } \zeta(\tau_0) + \dots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \dots + \zeta(\tau_k) - 1, \end{aligned}$$

for $k \geq 1$, and with the convention that $\zeta(\tau_0) = 0$. If there is a finite number of trees in the forest, say j , then we set $H_n(\mathcal{F}) = 0$, for $n \geq \zeta(\tau_0) + \dots + \zeta(\tau_j)$.

For two vertices u and v of a tree τ , the distance $d_\tau(u, v)$ is the number of edges of the unique elementary path from u to v . The height function may be presented in a natural way as the distance between the visited vertex and the root \emptyset , i.e. $H_n(\tau) = d_\tau(\emptyset, u(n))$. Then we may check the following relation

$$d_\tau(u(n), u(m)) = H_n(\tau) + H_m(\tau) - 2H_{k(n,m)}(\tau), \quad (2.1)$$

where $k(n, m)$ is the index of the last common ancestor of $u(n)$ and $u(m)$. It is not difficult to see that the height process of a tree (resp. a forest) allows us to recover the entire structure of this tree (resp. this forest). We say that it *codes* the genealogy of the tree or the forest. Although this process is natural and simple to define, its law is rather complicated to characterize. In particular, H is neither a Markov process nor a martingale.

The contour process gives another characterization of the tree which is easier to visualize. We suppose that the tree is embedded in a half-plane in such a way that edges have length one. Informally, we imagine the motion of a particle that starts at time 0 from the root of the tree and then explores the tree from the left to the right continuously along each edge of τ at unit speed until all edges have been explored and the particle has come back to the root. Note that if $u(n)$ is a leaf, then the particle goes to $u(n+1)$, taking the shortest way that consists first to move backward on the line of descent from $u(n)$ to their last common ancestor $u(n) \wedge u(n+1)$ and then to move forward along the single edge between $u(n) \wedge u(n+1)$ to $u(n+1)$. Since it is clear that each edge will be crossed twice, the total time needed to explore the tree is $2(\zeta(\tau) - 1)$. The value $C_t(\tau)$ of the contour function at time $t \in [0, 2(\zeta(\tau) - 1)]$ is the distance (on the continuous tree) between the position of the particle at time s and the root. More precisely, let us denote by $l_1 < l_2 < \dots < l_p$ the p leaves of τ listed in lexicographical order. The contour function $(C_t(\tau), 0 \leq t \leq 2(\zeta(\tau) - 1))$ is the piecewise linear continuous path with slope equal to +1 or -1, that takes successive local extremes with values: $0, |l_1|, |l_1 \wedge l_2|, |l_2|, \dots, |l_{p-1} \wedge l_p|, |l_p|$ and 0. Then we set $C_t(\tau) = 0$, for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. It is clear that $C(\tau)$ codes the genealogy of τ .

The contour process for a forest $\mathcal{F} = (\tau_k)$ is the concatenation of the processes $C(\tau_1), \dots, C(\tau_k), \dots$, i.e. for $k \geq 1$:

$$C_t(\mathcal{F}) = C_{t-2(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k), \quad \text{if } 2(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1})) \leq t \leq 2(\zeta(\tau_0)+\dots+\zeta(\tau_k)).$$

If there is a finite number of trees, say j , in the forest, we set $C_t(\mathcal{F}) = 0$, for $t \geq 2(\zeta(\tau_0) + \dots + \zeta(\tau_j))$. Note that for each tree τ_k , $[2(\zeta(\tau_k) - 1), 2\zeta(\tau_k)]$ is the only non-trivial subinterval of $[0, 2\zeta(\tau_k)]$ on which $C(\tau_k)$ vanishes. This convention ensures that the contour process $C(\mathcal{F})$ also codes the genealogy of the forest. However, it has no "good properties" in law either.

In order to define a coding process whose law can easily be described, most of the authors introduce the coding random walk $S(\tau)$ which is defined as follows:

$$S_0 = 0, \quad S_{n+1}(\tau) - S_n(\tau) = k_{u(n)}(\tau) - 1, \quad 0 \leq n \leq \zeta(\tau) - 1.$$

Here again it is not very difficult to see that the process $S(\tau)$ codes the genealogy of the tree τ . However, its construction requires a little bit more care than this of $H(\tau)$ or $C(\tau)$. For each n , $S_n(\tau)$ is the sum of all the younger brother of each of the ancestor of $u(n)$ including $u(n)$ itself. For a forest $\mathcal{F} = (\tau_k)$, the process $S(\mathcal{F})$ is the concatenation of $S(\tau_1), \dots, S(\tau_k), \dots$:

$$S_n(\mathcal{F}) = S_{n-(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k) - k + 1, \\ \text{if } \zeta(\tau_0) + \dots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \dots + \zeta(\tau_k).$$

If there is a finite number of trees j , then we set $S_n(\mathcal{F}) = S_{\zeta(\tau_0)+\dots+\zeta(\tau_j)}(\mathcal{F})$, for $n \geq \zeta(\tau_0)+\dots+\zeta(\tau_j)$. From the construction of $S(\tau_1)$ it appears that $S(\tau_1)$ is a random walk with initial value $S_0 = 0$ and step distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \dots$ which is killed when it first enters into the negative half-line. Hence, when the number of trees is infinite, $S(\mathcal{F})$ is a downward skip free random walk on \mathbb{Z} with the law described above.

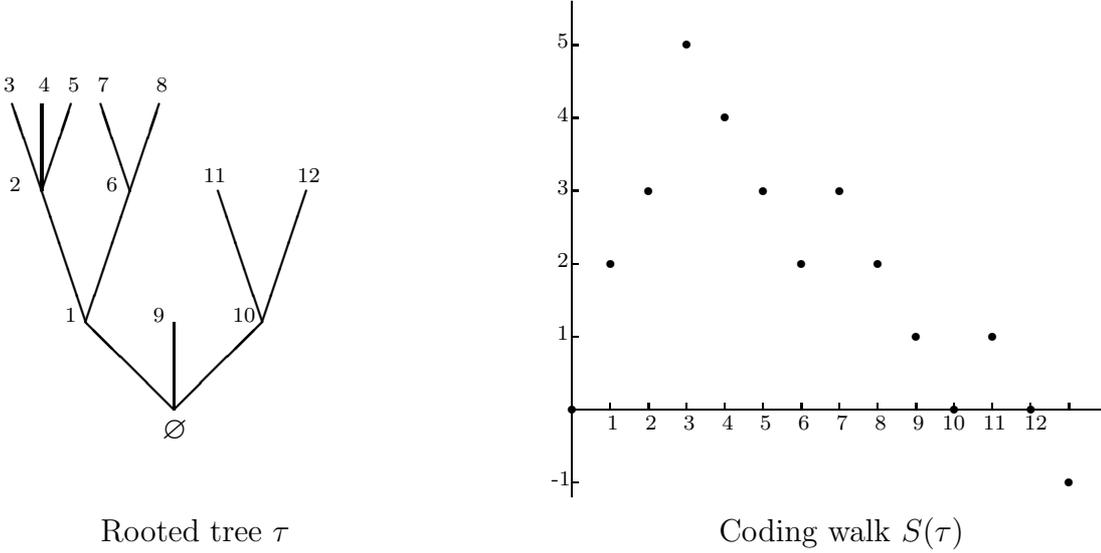


Figure 1

Let us denote $H(\mathcal{F})$, $C(\mathcal{F})$ and $S(\mathcal{F})$ respectively by H , C and S when no confusion is possible. In the sequel, we will have to use some path relationships between H , C and S which we recall now. Let us suppose that \mathcal{F} is infinite. It is established for instance in [9, 21] that

$$H_n = \text{card} \{0 \leq k \leq n-1 : S_k = \inf_{k \leq j \leq n} S_j\}. \quad (2.2)$$

This identity means that the height process at each time n can be interpreted as the amount of time that the random walk S spends at its future minimum before n . The following relationship between H and C is stated in [9]: set $K_n = 2n - H_n$, then

$$C_t = \begin{cases} (H_n - (t - K_n))^+, & \text{if } t \in [K_n, K_{n+1} - 1] \\ (H_{n+1} - (K_{n+1} - t))^+, & \text{if } [K_{n+1} - 1, K_{n+1}]. \end{cases} \quad (2.3)$$

For any integer $k \geq 1$, we denote by $\mathcal{F}^{k,n}$ a G-W forest with k trees conditioned to have n vertices, that is a forest with the same law as $\mathcal{F} = (\tau_1, \dots, \tau_k)$ under the conditional law $\mathbb{Q}_\mu(\cdot | \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$. The starting point of our work is the observation that $\mathcal{F}^{k,n}$ can be coded by a downward skip free random walk conditioned to first reach $-k$ at time n . An interpretation of this result may be found in [22], Lemma 6.3 for instance.

Proposition 1. Let $\mathcal{F} = (\tau_j)$ be an infinite forest with offspring distribution μ and S , H and C be respectively its coding walk, its height process and its contour process. Let W be a random walk defined on a probability space (Ω, \mathcal{F}, P) with the same law as S . We define $T_i^W = \inf\{j : W_j = -i\}$, for $i \geq 1$. Take k and n such that $P(T_k^W = n) > 0$. Then under the conditional law $\mathbb{Q}_\mu(\cdot | \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$,

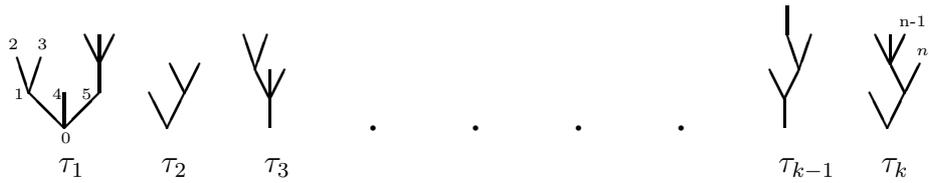
(1) The process $(S_j, 0 \leq j \leq \zeta(\tau_1) + \dots + \zeta(\tau_k))$ has the same law as $(W_j, 0 \leq j \leq T_k^W)$.

Moreover, define the processes $H_n^W = \text{card}\{k \in \{0, \dots, n-1\} : W_k = \inf_{k \leq j \leq n} W_j\}$ and C^W using the height process H^W as in (2.3), then

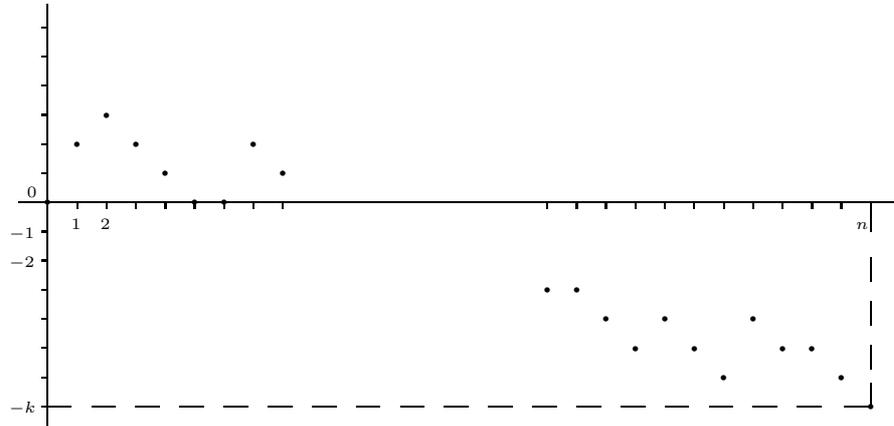
(2) the process $(H_j, 0 \leq j \leq \zeta(\tau_1) + \dots + \zeta(\tau_k))$ has the same law as the process $(H_j^W, 0 \leq j \leq T_k^W)$.

(3) the process $(C_t, 0 \leq t \leq 2(\zeta(\tau_1) + \dots + \zeta(\tau_k)))$ has the same law as the process $(C_t^W, 0 \leq t \leq 2T_k^W)$.

It is also straightforward that the identities in law involving separately the processes H , S and C in the above proposition also hold for the triple (H, S, C) . In the figure below, we have represented an occurrence of the forest $\mathcal{F}^{k,n}$ and its associated coding first passage bridge.



Conditioned random forest: k trees, n vertices.



Coding first passage bridge

Figure 2

In Section 4, we will present a continuous time version of this result, but before we need to introduce the continuous time setting of Lévy trees and forests.

3 Coding real trees and forests

Discrete trees may be considered in an obvious way as compact metric spaces with no loops. Such metric spaces are special cases of \mathbb{R} -trees which are defined hereafter. Similarly to the discrete case, an \mathbb{R} -forest is any collection of \mathbb{R} -trees. In this section we keep the same notations as in Duquesne and Le Gall's articles [9] and [11]. The following formal definition of \mathbb{R} -trees is now classical and originates from T -theory. It may be found for instance in [7].

Definition 2. *A metric space (\mathcal{T}, d) is an \mathbb{R} -tree if for every $\sigma_1, \sigma_2 \in \mathcal{T}$,*

1. *There is a unique map f_{σ_1, σ_2} from $[0, d(\sigma_1, \sigma_2)]$ into \mathcal{T} such that $f_{\sigma_1, \sigma_2}(0) = \sigma_1$ and $f_{\sigma_1, \sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2$.*
2. *If g is a continuous injective map from $[0, 1]$ into \mathcal{T} such that $g(0) = \sigma_1$ and $g(1) = \sigma_2$, we have*

$$g([0, 1]) = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)]).$$

A rooted \mathbb{R} -tree is an \mathbb{R} -tree (\mathcal{T}, d) with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root. An \mathbb{R} -forest is any collection of rooted \mathbb{R} -trees: $\mathcal{F} = \{(\mathcal{T}_i, d_i), i \in \mathcal{I}\}$.

A construction of some particular cases of such metric spaces has been given by Aldous [1] and is described in [11] in a more general setting. Let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous function with compact support, such that $f(0) = 0$. For $0 \leq s \leq t$, we define

$$d_f(s, t) = f(s) + f(t) - 2 \inf_{u \in [s, t]} f(u) \tag{3.4}$$

and the equivalence relation by

$$s \sim t \text{ if and only if } d_f(s, t) = 0.$$

(Note that $d_f(s, t) = 0$ if and only if $f(s) = f(t) = \inf_{u \in [s, t]} f(u)$.) Then the projection of d_f on the quotient space

$$\mathcal{T}_f = [0, \infty) / \sim$$

defines a distance. This distance will also be denoted by d_f .

Theorem 1. *The metric space (\mathcal{T}_f, d_f) is a compact \mathbb{R} -tree.*

Denote by $p_f : [0, \infty) \rightarrow \mathcal{T}_f$ the canonical projection. The vertex $\rho = p_f(0)$ will be chosen as the root of \mathcal{T}_f . It has recently been proved by Duquesne [8] that any \mathbb{R} -tree (satisfying some rather weak assumptions) may be represented as (\mathcal{T}_f, d_f) where f is a left continuous function with right limits and without positive jumps.

When no confusion is possible with the discrete case, the space of \mathbb{R} -trees will also be denoted by \mathbb{T} . It is endowed with the Gromov-Hausdorff distance, d_{GH} which we briefly recall now. For a metric space (E, δ) and K, K' two subspaces of E , $\delta_{\text{Haus}}(K, K')$ will denote the Hausdorff distance between K and K' . Then we define the distance between \mathcal{T} and \mathcal{T}' by:

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \inf (\delta_{\text{Haus}}(\varphi(\mathcal{T}), \varphi'(\mathcal{T}')) \vee \delta(\varphi(\rho), \varphi'(\rho'))),$$

where the infimum is taken over all isometric embeddings $\varphi : \mathcal{T} \rightarrow E$ and $\varphi' : \mathcal{T}' \rightarrow E$ of \mathcal{T} and \mathcal{T}' into a common metric space (E, δ) . We refer to Chapter 3 of Evans [12] and the references therein for a complete description of the Gromov-Hausdorff topology. It is important to note that the space (\mathbb{T}, d_{GH}) is complete and separable, see for instance Theorem 3.23 of [12] or [13].

In the remainder of this section, we will recall from [11] the definition of Lévy trees and we state this of Lévy forests. Let (\mathbb{P}_x) , $x \in \mathbb{R}$ be a sequence of probability measures on the Skorokhod space \mathbb{D} of càdlàg paths from $[0, \infty)$ to \mathbb{R} such that for each $x \in \mathbb{R}$, the canonical process X is a Lévy process with no negative jumps. Set $\mathbb{P} = \mathbb{P}_0$, so \mathbb{P}_x is the law of $X + x$ under \mathbb{P} . We suppose that the characteristic exponent ψ of X (i.e. $\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)}$, $\lambda \in \mathbb{R}$) satisfies the following condition:

$$\int_1^\infty \frac{du}{\psi(u)} < \infty. \quad (3.5)$$

By analogy with the discrete case, the continuous time height process \bar{H} is the measure (in a sense which is to be defined) of the set $\{s \leq t : X_s = \inf_{s \leq r \leq t} X_r\}$. A rigorous meaning to this measure is given by the following result due to Le Jan and Le Gall [21], see also [9]. Define $I_t^s = \inf_{s \leq u \leq t} X_u$. There is a sequence of positive real numbers (ε_k) which decreases to 0 such that for any t , the limit

$$\bar{H}_t \stackrel{\text{(def)}}{=} \lim_{k \rightarrow +\infty} \frac{1}{\varepsilon_k} \int_0^t \mathbb{1}_{\{X_s - I_t^s < \varepsilon_k\}} ds \quad (3.6)$$

exists a.s. It is also proved in [21] that under assumption (3.5), \bar{H} is a continuous process, so that each of its positive excursion codes a real tree in the sense of Aldous. We easily deduce from this definition that the height process \bar{H} is a functional of the Lévy process reflected at its minimum, i.e. $X - I$, where $I := I^0$. In particular, when $\alpha = 2$, \bar{H} is equal in law to the reflected process multiplied by a constant. It is well known that $X - I$ is a strong Markov process. Moreover, under our assumptions, 0 is regular for itself for this process and we can check that the process $-I$ is a local time at level 0. We denote by N the corresponding Itô measure of the excursions away from 0.

In order to define the Lévy forest, we need to introduce the local times of the height process \bar{H} . It is proved in [9] that for any level $a \geq 0$, there exists a continuous increasing process $(L_t^a, t \geq 0)$ which is defined by the approximation:

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left(\sup_{0 \leq s \leq t} \left| \frac{1}{\varepsilon} \int_0^s du \mathbb{1}_{\{a < \bar{H}_u \leq a + \varepsilon\}} - L_s^a \right| \right) = 0. \quad (3.7)$$

The support of the measure dL_t^a is contained in the set $\{t \geq 0 : \bar{H}_t = a\}$ and we readily check that $L^0 = -I$. Then we may define the Poisson point process of the excursions away from 0 of the process \bar{H} as follows. Let $T_u = \inf\{t : -I_t \geq u\}$ be the right continuous inverse of the local time at 0 of the reflected process $X - I$ (or equivalently of H). The time T_u corresponds to the first passage time of X below $-u$. Set $T_{0-} = 0$ and for all $u \geq 0$,

$$e_u(v) = \begin{cases} \bar{H}_{T_{u-}+v}, & \text{if } 0 \leq v \leq T_u - T_{u-} \\ 0, & \text{if } v > T_u - T_{u-} \end{cases}.$$

For each $u \geq 0$, we may define the tree $(\mathcal{T}_{e_u}, d_{e_u})$ under \mathbb{P} as in the beginning of this section. We easily deduce from the Markov property of $X - I$ that under the probability measure \mathbb{P} , the process $\{(\mathcal{T}_{e_u}, d_{e_u}), u \geq 0\}$ is a Poisson point process whose characteristic measure is the law of the random real tree $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ under N . By analogy to the discrete case, this Poisson point process, as a \mathbb{T} -valued process, provides a natural definition for the Lévy forest.

Definition 3. *The Lévy tree is the real tree $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ coded by the function \bar{H} under the measure N . We denote by $\Theta(d\mathcal{T})$ the σ -finite measure on \mathbb{T} which is the law of the Lévy tree $\mathcal{T}_{\bar{H}}$ under N . The Lévy forest $\mathcal{F}_{\bar{H}}$ is the Poisson point process*

$$(\mathcal{F}_{\bar{H}}(u), u \geq 0) \stackrel{(\text{def})}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), u \geq 0\}$$

which has for characteristic measure $\Theta(d\mathcal{T})$ under \mathbb{P} . For each $s > 0$, the process $\mathcal{F}_{\bar{H}}^s \stackrel{(\text{def})}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), 0 \leq u \leq s\}$ under \mathbb{P} will be called the Lévy forest of size s .

Such a definition of a Lévy forest has already been introduced in [22], Proposition 7.8 in the Brownian setting. However in this work, it is observed that the Brownian forest may also simply be defined as the real tree coded by the function \bar{H} under law \mathbb{P} . We also refer to [23] where the Brownian forest is understood in this way. Similarly, the Lévy forest with size s may be defined as the compact real tree coded by the continuous function with compact support $(\bar{H}_u, 0 \leq u \leq T_s)$ under law \mathbb{P} . These definitions are more natural when considering convergence of sequences of real forests and we will make appeal to them in section 5, see Corollary 1.

We will simply denote the Lévy tree and the Lévy forest respectively by $\mathcal{T}_{\bar{H}}$, $\mathcal{F}_{\bar{H}}$ or $\mathcal{F}_{\bar{H}}^s$, the corresponding distances being implicit. When X is stable, condition (3.5) is satisfied if and only if its index α satisfies $\alpha \in (1, 2)$. We may check, as a consequence of (3.6), that \bar{H} is a self-similar process with index $\alpha/(\alpha - 1)$, i.e.:

$$(\bar{H}_t, t \geq 0) \stackrel{(d)}{=} (k^{(\alpha-1)/\alpha} \bar{H}_{kt}, t \geq 0), \quad \text{for all } k > 0.$$

In this case, the Lévy tree $\mathcal{T}_{\bar{H}}$ associated to the stable mechanism is called the α -stable Lévy tree and its law is denoted by $\Theta_\alpha(d\mathcal{T})$. This random metric space also inherits from X a scaling property which may be stated as follows: for any $a > 0$, we denote by $a\mathcal{T}_{\bar{H}}$ the Lévy tree $\mathcal{T}_{\bar{H}}$ endowed with the distance $ad_{\bar{H}}$, i.e.

$$a\mathcal{T}_{\bar{H}} \stackrel{(\text{def})}{=} (\mathcal{T}_{\bar{H}}, ad_{\bar{H}}). \tag{3.8}$$

Then the law of $a\mathcal{T}_{\bar{H}}$ under $\Theta_\alpha(d\mathcal{T})$ is $a^{\frac{1}{\alpha-1}} \Theta_\alpha(d\mathcal{T})$. This property is stated in [20] Proposition 4.3 and [10] where other fractal properties of stable trees are considered.

4 Construction of the conditioned Lévy forest

In this section we present the continuous analogue of the forest $\mathcal{F}^{k,n}$ introduced in section 2. In particular, we define the total mass of the Lévy forest of a given size s . Then we define the Lévy forest of size s conditioned by its total mass. In the stable case, we give a construction of this conditioned forest from the unconditioned forest.

We begin with the definition of the measure $\ell^{a,u}$ which represents a local time at level $a > 0$ for the Lévy tree \mathcal{T}_{e_u} . For all $a > 0$, $u \geq 0$ and for every bounded and continuous function φ on \mathcal{T}_{e_u} , the finite measure $\ell^{a,u}$ is defined by:

$$\langle \ell^{a,u}, \varphi \rangle = \int_0^{T_u - T_{u-}} dL_{T_u+v}^a \varphi(p_{e_u}(v)), \quad (4.9)$$

where we recall from the previous section that p_{e_u} is the canonical projection from $[0, \infty)$ onto \mathcal{T}_{e_u} for the equivalence relation \sim and (L_u^a) is the local time at level a of \bar{H} . Then the mass measure of the Lévy tree \mathcal{T}_{e_u} is

$$\mathbf{m}_u = \int_0^\infty da \ell^{a,u} \quad (4.10)$$

and the total mass of the tree is $\mathbf{m}_u(\mathcal{T}_{e_u})$. Now we fix $s > 0$; the total mass of the forest of size s , $\mathcal{F}_{\bar{H}}^s$ is naturally given by

$$\mathbf{M}_s = \sum_{0 \leq u \leq s} \mathbf{m}_u(\mathcal{T}_{e_u}).$$

Proposition 2. \mathbb{P} -almost surely $T_s = \mathbf{M}_s$.

Proof. It follows from the definitions (4.9) and (4.10) that for each tree \mathcal{T}_{e_u} , the mass measure \mathbf{m}_u coincides with the image of the Lebesgue measure on $[0, T_u - T_{u-}]$ under the mapping $v \mapsto p_{e_u}(v)$. Thus, the total mass $\mathbf{m}_u(\mathcal{T}_{e_u})$ of each tree \mathcal{T}_{e_u} is $T_u - T_{u-}$. This implies the result. \blacksquare

Then we will construct processes which encode the genealogy of the Lévy forest of size s conditioned to have a mass equal to $t > 0$. From the analogy with the discrete case in Proposition 1, the natural candidates may informally be defined as:

$$\begin{aligned} X^{br} &\stackrel{(\text{def})}{=} [(X_u, 0 \leq u \leq T_s) | T_s = t] \\ \bar{H}^{br} &\stackrel{(\text{def})}{=} [(\bar{H}_u, 0 \leq u \leq T_s) | T_s = t]. \end{aligned}$$

When X is the Brownian motion, the process X^{br} is called the *first passage bridge*, see [3]. In order to give a proper definition in the general case, we need the additional assumption:

The semigroup of (X, \mathbb{P}) is absolutely continuous with respect to the Lebesgue measure.

Then denote by $p_t(\cdot)$ the density of the semigroup of X , by $\mathcal{G}_u^X \stackrel{(\text{def})}{=} \sigma\{X_v, v \leq u\}$, $u \geq 0$ the σ -field generated by X and set $\hat{p}_t(x) = p_t(-x)$.

Lemma 1. *The probability measure which is defined on each \mathcal{G}_u^X by*

$$\mathbb{P}(X^{br} \in \Lambda_u) = \mathbb{E} \left(\mathbb{1}_{\{X \in \Lambda_u, u < T_s\}} \frac{t(s + X_u) \hat{p}_{t-u}(s + X_u)}{s(t - u) \hat{p}_t(s)} \right), \quad u < t, \quad \Lambda_u \in \mathcal{G}_u^X, \quad (4.11)$$

is a regular version of the conditional law of $(X_u, 0 \leq u \leq T_s)$ given $T_s = t$, in the sense that for all $u > 0$, for λ -a.e. $s > 0$ and λ -a.e. $t > u$,

$$\mathbb{P}(X^{br} \in \Lambda_u) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(X \in \Lambda_u \mid |T_s - t| < \varepsilon),$$

where λ is the Lebesgue measure.

Proof. Let $u < t$, $\Lambda_u \in \mathcal{G}_u^X$ and $\varepsilon < t - u$. From the Markov property, we may write

$$\begin{aligned} \mathbb{P}(X \in \Lambda_u \mid |T_s - t| < \varepsilon) &= \mathbb{E} \left(\mathbb{1}_{\{X \in \Lambda_u\}} \frac{\mathbb{1}_{\{|T_s - t| < \varepsilon\}}}{\mathbb{P}(|T_s - t| < \varepsilon)} \right) \\ &= \mathbb{E} \left(\mathbb{1}_{\{X \in \Lambda_u, u < T_s\}} \frac{\mathbb{P}_{X_u}(|T_s - (t - u)| < \varepsilon)}{\mathbb{P}(|T_s - t| < \varepsilon)} \right). \end{aligned} \quad (4.12)$$

On the other hand, from Corollary VII.3 in [2] one has,

$$t\mathbb{P}(T_s \in dt) ds = s\hat{p}_t(s) dt ds. \quad (4.13)$$

Hence, for all $x \in \mathbb{R}$, for all $u > 0$, for λ -a.e. $s > 0$ and λ -a.e. $t > u$,

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_x(|T_s - (t - u)| < \varepsilon)}{\mathbb{P}(|T_s - t| < \varepsilon)} = \frac{t(s + x) \hat{p}_{t-u}(s + x)}{s(t - u) \hat{p}_t(s)}.$$

Moreover we can check from (4.13) that $\mathbb{E} \left(\frac{t(s + X_u) \hat{p}_{t-u}(s + X_u)}{s(t - u) \hat{p}_t(s)} \right) < +\infty$ for λ -a.e. t , so the result follows from (4.12) and Fatou's lemma. \blacksquare

We may now construct a height process \bar{H}^{br} from the path of the first passage bridge X^{br} exactly as \bar{H} is constructed from X in (3.6) or in Definition 1.2.1 of [9] and check that the law of \bar{H}^{br} is a regular version of the conditional law of $(\bar{H}_u, 0 \leq u \leq T_s)$ given $T_s = t$. Call $(e_u^{s,t}, 0 \leq u \leq s)$ the excursion process of \bar{H}^{br} , that is in particular

$$(e_u^{s,t}, 0 \leq u \leq s) \text{ has the same law as } (e_u, 0 \leq u \leq s) \text{ given } T_s = t.$$

The following proposition is a straightforward consequence of the above definition and Proposition 2.

Proposition 3. *The law of the process $\{(\mathcal{T}_{e_u^{s,t}}, d_{e_u^{s,t}}), 0 \leq u \leq s\}$ is a regular version of the law of the forest of size s , $\mathcal{F}_{\bar{H}}^s$ given $\mathbf{M}_s = t$.*

We will denote by $(\mathcal{F}_{\bar{H}}^{s,t}(u), 0 \leq u \leq s)$ a process with values in \mathbb{T} whose law under \mathbb{P} is this of the Lévy forest of size s conditioned by $\mathbf{M}_s = t$, i.e. conditioned to have a mass equal to t .

In the remainder of this section, we will consider the case when the driving Lévy process is stable. We suppose that its index α belongs to $(1, 2]$ so that condition (3.5) is satisfied. We will give a pathwise construction of the processes (X^{br}, \bar{H}^{br}) from the path of the original processes (X, \bar{H}) . This result leads to the following realization of the Lévy forest of size s conditioned by its mass. From now on, with no loss of generality, we suppose that $t = 1$.

Theorem 2. Define $g = \sup\{u \leq 1 : T_{u^{1/\alpha}} = s \cdot u\}$.

(1) \mathbb{P} -almost surely,

$$0 < g < 1.$$

(2) Under \mathbb{P} , the rescaled process

$$(g^{(1-\alpha)/\alpha} \bar{H}(gu), 0 \leq u \leq 1) \tag{4.14}$$

has the same law as \bar{H}^{br} and is independent of g .

(3) The forest $\mathcal{F}_{\bar{H}}^{s,1}$ of size s and mass 1 may be constructed from the rescaled process defined in (4.14), i.e. if we denote by $u \mapsto \epsilon_u \stackrel{(def)}{=} (g^{(1-\alpha)/\alpha} e_u(gv), v \geq 0)$ its process of excursions away from 0, then under \mathbb{P} , $\mathcal{F}_{\bar{H}}^{s,1} \stackrel{(d)}{=} \{(\mathcal{T}_{\epsilon_u}, d_{\epsilon_u}), 0 \leq u \leq s\}$.

Proof. The process $T_u = \inf\{v : I_v \leq -u\}$ is a stable subordinator with index $1/\alpha$. Therefore,

$$T_u < su^\alpha, \quad \text{i.o. as } u \downarrow 0 \quad \text{and} \quad T_u > su^\alpha, \quad \text{i.o. as } u \downarrow 0.$$

Indeed, if $u_n \downarrow 0$ then $\mathbb{P}(T_{u_n} < su_n^\alpha) = \mathbb{P}(T_1 < s) > 0$, so that $\mathbb{P}(\limsup_n \{T_{u_n} < su_n^\alpha\}) \geq \mathbb{P}(T_1 < s) > 0$. But T satisfies Blumenthal 0-1 law, so this probability is 1. The same arguments prove that $\mathbb{P}(\limsup_n \{T_{u_n} > su_n^\alpha\}) = 1$ for any sequence $u_n \downarrow 0$. Since T has only positive jumps, we deduce that $T_u = su^\alpha$ infinitely often as u tends to 0, so we have proved the first part of the theorem.

The rest of the proof is a consequence of the following lemma.

Lemma 2. The first passage bridge X^{br} enjoys the following path construction:

$$X^{br} \stackrel{(d)}{=} (g^{-1/\alpha} X(gu), 0 \leq u \leq 1).$$

Moreover, the process $(g^{-1/\alpha} X(gu), 0 \leq u \leq 1)$ is independent of g .

Proof. First note that for any $t > 0$ the bivariate random variable (X_t, I_t) under \mathbb{P} is absolutely continuous with respect to the Lebesgue measure and there is a version of its density which is continuous. Indeed from the Markov property and (4.13), one has for all $x \in \mathbb{R}$ and $y \geq 0$,

$$\begin{aligned} \mathbb{P}(I_t \leq y | X_t = x) &= \mathbb{E} \left(\mathbb{1}_{\{T_y \leq t\}} \frac{p_{t-T_y}(x-y)}{p_t(0)} \right) \\ &= \int_0^t \frac{y}{s} \hat{p}_s(y) \frac{p_{t-s}(x-y)}{p_t(0)} ds. \end{aligned}$$

Looking at the expressions of $\hat{p}_t(x)$ and $p_t(x)$ obtained from the Fourier inverse of the characteristic exponent of X and $-X$ respectively, we see that these functions are continuously differentiable and that their derivatives are continuous in t . It allows us to conclude.

Now let us consider the two dimensional self-similar strong Markov process $Y \stackrel{(\text{def})}{=} (X, I)$ with state space $\{(x, y) \in \mathbb{R}^2 : y \leq x\}$. From our preceding remark, the semi-group $q_t((x, y), (dx', dy')) = \mathbb{P}(X_t + x \in dx', y \wedge (I_t + x) \in dy')$ of Y is absolutely continuous with respect to the Lebesgue measure and there is a version of its density which is continuous. Denote by $q_t((x, y), (x', y'))$ this version. We derive from (4.13) that for all $-s \leq x$,

$$q_t((x, y), (-s, -s)) = \mathbb{1}_{\{y \geq -s\}} \frac{1}{t} \hat{p}_t(s + x). \quad (4.15)$$

Then we may apply a result due to Fitzsimmons, Pitman and Yor [15] which asserts that the inhomogenous Markov process on $[0, t]$, whose law is defined by

$$\mathbb{E} \left(H(Y_u, v \leq u) \frac{q_{t-u}(Y_u, (x', y'))}{q_t((x, y), (x', y'))} \mid Y_0 = (x, y) \right), \quad 0 \leq u < t, \quad (4.16)$$

where H is a measurable functional on $C([0, u], \mathbb{R}^2)$, is a regular version of the conditional law of $(Y_v, 0 \leq v \leq t)$ given $Y_t = (x', y')$, under $\mathbb{P}(\cdot \mid Y_0 = (x, y))$. This law is called the law of the bridge of Y from (x, y) to (x', y') with length t . Then from (4.15), the law which is defined in (4.16), when specifying it on the first coordinate and for $(x, y) = (0, 0)$ and $(x', y') = (-s, -s)$, corresponds to the law of the first passage bridge which is defined in (4.11).

It remains to apply another result which may also be found in [15]: observe that g is a backward time for Y in the sense of [15]. Indeed g we may check that $g = \sup\{u \leq 1 : X_u = -su^{1/\alpha}, X_u = I_u\}$, so that for all $u > 0$, $\{g > u\} \in \sigma(Y_v : v \geq u)$. Then from Corollary 3 in [15], conditionally on g , the process $(Y_u, 0 \leq u \leq g)$ under $\mathbb{P}(\cdot \mid Y_0 = (0, 0))$ has the law of a bridge from $(0, 0)$ to Y_g with length g . (This result has been obtained and studied in a greater generality in [6].) But from the definition of g , we have $Y_g = (-sg^{1/\alpha}, -sg^{1/\alpha})$, so from the self-similarity of Y , under \mathbb{P} the process

$$(g^{-1/\alpha} Y(g \cdot u), 0 \leq u \leq 1)$$

has the law of the bridge of Y from $(0, 0)$ to $(-s, -s)$ with length 1. The lemma follows by specifying this result on the first coordinate. \blacksquare

The second part of the theorem is a consequence of Lemma 2, the construction of \bar{H}^{br} from X^{br} and the scaling property of \bar{H} . The third part follows from the definition of the conditioned forest $\mathcal{F}_{\bar{H}}^{s,1}$ in Proposition 2 and the second part of this theorem. \blacksquare

5 Invariance principles

We know from Lamperti that the only possible limits of sequences of re-scaled G-W processes are continuous state branching processes. Then a question which arises is: when can we say that the whole genealogy of the tree or the forest converges? In particular, do the height process, the contour process and the coding walk converge after a suitable re-scaling? This question has now been completely solved by Duquesne and Le Gall [9]. Then one may ask the same for the trees or forests conditioned by their mass. In [8], Duquesne proved that when the law ν is in the domain of attraction of a stable law, the height process, the contour process and the coding excursion of the corresponding G-W tree converge in law in the Skorokhod space of càdlàg paths. This work generalizes Aldous' result [1] which concerns the Brownian case. In this section we will prove that in the stable case, an invariance principle also holds for sequences of G-W forests conditioned by their mass.

Recall from section 2 that for an offspring distribution μ we have set $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, \dots$. We make the following assumption:

$$(H) \quad \begin{cases} \mu \text{ is aperiodic and there is an increasing sequence } (a_n)_{n \geq 0} \\ \text{such that } a_n \rightarrow +\infty \text{ and } S_n/a_n \text{ converges in law as } n \rightarrow +\infty \\ \text{toward the law of a non-degenerated r.v. } \theta. \end{cases}$$

Note that we are necessarily in the critical case, i.e. $\sum_k k\mu(k) = 1$, and that the law of θ is stable. Moreover, since $\nu(-\infty, -1) = 0$, the support of the Lévy measure of θ is $[0, \infty)$ and its index α is such that $1 < \alpha \leq 2$. Also (a_n) is a regularly varying sequence with index α . Under hypothesis (H), it has been proved by Grimvall [16] that if Z is the G-W process associated to a tree or a forest with offspring distribution μ , then

$$\left(\frac{1}{a_n} Z_{[nt/a_n]}, t \geq 0 \right) \Rightarrow (\bar{Z}_t, t \geq 0), \quad \text{as } n \rightarrow +\infty,$$

where $(\bar{Z}_t, t \geq 0)$ is a continuous state branching process. Here and in the sequel, \Rightarrow will stand for the weak convergence in the Skorokhod space of càdlàg trajectories. Recall from section 2 the definition of the discrete process (S, H, C) . Under the same hypothesis, Duquesne and Le Gall have proved in [9], Corollary 2.5.1 that

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt} \right), t \geq 0 \right] \Rightarrow [(X_t, \bar{H}_t, \bar{H}_t), t \geq 0], \quad \text{as } n \rightarrow +\infty, \quad (5.17)$$

where X is a stable Lévy process with law θ and \bar{H} is the associated height process, as defined in section 3.

Again we fix a real $s > 0$ and we consider a sequence of positive integers (k_n) such that

$$\frac{k_n}{a_n} \rightarrow s, \quad \text{as } n \rightarrow +\infty. \quad (5.18)$$

Recall the notations of section 2. For any $n \geq 1$, let $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ be the process whose law is this of

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt} \right), 0 \leq t \leq 1 \right],$$

under $\mathbb{Q}_\mu(\cdot | \zeta(\tau_1) + \dots + \zeta(\tau_{k_n}) = n)$. Note that we could also define this three dimensional process over the whole halfline $[0, \infty)$, rather than on $[0, 1]$. However, from the definitions in section 2, $\bar{H}^{br,n}$ and $C^{br,n}$ would simply vanish over $[1, \infty)$ and $X^{br,n}$ would be constant. Here is the conditional version of the invariance principle that we have recalled in (5.17).

Theorem 3. *As n tends to $+\infty$, we have*

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n}) \Longrightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br}).$$

In order to give a sense to the convergence of the Lévy forest, we may consider the trees $\mathcal{T}^{br,n}$ and \mathcal{T}^{br} which are coded respectively by the continuous processes with compact support, $C_u^{br,n}$ and \bar{H}_u^{br} , in the sense given at the beginning of section 3 (here we suppose that these processes are defined on $[1, \infty)$ and both equal to 0 on this interval). Roughly speaking the trees $\mathcal{T}^{br,n}$ and \mathcal{T}^{br} are obtained from the original (conditioned) forests by rooting all the trees of these forests at a same root.

Corollary 1. *The sequence of trees $\mathcal{T}^{br,n}$ converges weakly in the space \mathbb{T} endowed with the Gromov-Hausdorff topology towards \mathcal{T}^{br} .*

Proof. This results is a consequence of the weak convergence of the contour function $C^{br,n}$ toward \bar{H}^{br} and the inequality

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g - g'\|,$$

which is proved in [11], see Lemma 2.3. (We recall that d_{GH} the Gromov-Hausdorff distance which has been defined in section 3.) ■

A first step for the proof of Theorem 3 is to obtain the weak convergence of $(X^{br,n}, \bar{H}^{br,n})$ restricted to the Skorokhod space $\mathbb{D}([0, t])$ for any $t < 1$. Then we will derive the convergence on $\mathbb{D}([0, 1])$ from an argument of cyclic exchangeability. The convergence of the third coordinate $C^{br,n}$ is a consequence of its particular expression as a functional of the process $\bar{H}^{br,n}$. In the remainder of the proof, we suppose that S is defined on the same probability space as X and has step distribution ν under \mathbb{P} . Define also $T_k = \inf\{i : S_i = -k\}$, for all integers $k \geq 0$. Hence the process $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ has the same law as

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt} \right), 0 \leq t \leq 1 \right],$$

under the conditional probability $\mathbb{P}(\cdot | T_{k_n} = n)$.

Lemma 3. *For any $t < 1$, as n tends to $+\infty$, we have*

$$[(X_u^{br,n}, \bar{H}_u^{br,n}), 0 \leq u \leq t] \Longrightarrow [(X_u^{br}, \bar{H}_u^{br}), 0 \leq u \leq t].$$

Proof. From Feller's combinatorial lemma, see [14], we have $\mathbb{P}(T_k = n) = \frac{k}{n}\mathbb{P}(S_n = -k)$, for all $n \geq 1, k \geq 0$. Let F be any bounded and continuous functional on $\mathbb{D}([0, t])$. By the Markov property at time $[nt]$,

$$\begin{aligned}
& \mathbb{E}[F(X_u^{br,n}, \bar{H}_u^{br,n}; 0 \leq u \leq t)] = \mathbb{E}\left[F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right) \mid T_{k_n} = n\right] \\
&= \mathbb{E}\left(\mathbb{I}_{\{[nt] \leq T_{k_n}\}} \frac{\mathbb{P}_{S_{[nt]}}(T_{k_n} = n - [nt])}{\mathbb{P}(T_{k_n} = n)} \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right)\right) \\
&= \mathbb{E}\left(\mathbb{I}_{\{\frac{1}{a_n}S_{[nt]} \geq -\frac{k_n}{a_n}\}} \frac{n(k_n + S_{[nt]})}{k_n(n - [nt])} \frac{\mathbb{P}_{S_{[nt]}}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)} \right. \\
&\quad \left. \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right)\right). \tag{5.19}
\end{aligned}$$

where $\underline{S}_k = \inf_{i \leq k} S_i$. To simplify the computations in the remainder of this proof, we set $P^{(n)}$ for the law of the process $\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; u \geq 0\right)$ and P will stand for the law of the process $(X_u, \bar{H}_u; u \geq 0)$. Then $Y = (Y^1, Y^2)$ is the canonical process of the coordinates on the Skorokhod space \mathbb{D}^2 of càdlàg paths from $[0, \infty)$ into \mathbb{R}^2 . We will also use special notations for the densities introduced in (4.11) and (5.19):

$$\begin{aligned}
D_t &= \mathbb{I}_{\{\underline{Y}_t^1 \geq -s\}} \frac{s + Y_t^1}{s(1-t)} \frac{\hat{p}_{1-t}(Y_t^1 + s)}{\hat{p}_1(s)}, \quad \text{and} \\
D_t^{(n)} &= \mathbb{I}_{\{\underline{Y}_{[nt]}^1 \geq -\frac{k_n}{a_n}\}} \frac{n(k_n + a_n Y_{[nt]}^1)}{k_n(n - [nt])} \frac{\mathbb{P}_{a_n Y_{[nt]}^1}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)},
\end{aligned}$$

where $\underline{Y}_t^1 = \inf_{u \leq t} Y_u^1$. Put also F_t for $F(Y_u, 0 \leq u \leq t)$. To obtain our result, we have to prove that

$$\lim_{n \rightarrow +\infty} |E^{(n)}(F_t D_t^{(n)}) - E(F_t D_t)| = 0. \tag{5.20}$$

Let $M > 0$ and set $I_M(x) \stackrel{\text{def}}{=} \mathbb{I}_{[-s, M]}(x)$. By writing

$$E^{(n)}(F_t D_t^{(n)}) = E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) + E^{(n)}(F_t D_t^{(n)} (1 - I_M(Y_t^1)))$$

and by doing the same for $E(F_t D_t)$, we have the following upper bound for the term in (5.20)

$$\begin{aligned}
|E^{(n)}(F_t D_t^{(n)}) - E(F_t D_t)| &\leq |E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))| \\
&\quad + C E^{(n)}(D_t^{(n)} (1 - I_M(Y_t^1))) + C E(D_t (1 - I_M(Y_t^1))),
\end{aligned}$$

where C is an upper bound for the functional F . But since D_t and $D_t^{(n)}$ are densities, $E^{(n)}(D_t^{(n)}) = 1$ and $E(D_t) = 1$, hence

$$\begin{aligned}
|E^{(n)}(F_t D_t^{(n)}) - E(F_t D_t)| &\leq |E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))| \\
&\quad + C[1 - E^{(n)}(D_t^{(n)} I_M(Y_t^1))] + C[1 - E(D_t I_M(Y_t^1))]. \tag{5.21}
\end{aligned}$$

Now it remains to prove that the first term of the right hand side of the inequality (5.21) tends to 0, i.e.

$$|E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))| \rightarrow 0, \quad (5.22)$$

as $n \rightarrow +\infty$. Indeed, suppose that (5.22) holds, then by taking $F_t \equiv 1$, we see that the second term of the right hand side of (5.21) converges towards the third one. Moreover, $E(D_t I_M(Y_t^1))$ tends to 1 as M goes to $+\infty$. Therefore the second and the third terms in (5.21) tend to 0 as n and M go to $+\infty$.

Let us prove (5.22). From the triangle inequality and the expression of the densities D_t and $D_t^{(n)}$, we have

$$\begin{aligned} |E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))| &\leq \sup_{x \in [-s, M]} |g_n(x) - g(x)| + \\ &|E^{(n)}(F_t D_t I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))|, \end{aligned} \quad (5.23)$$

where $g_n(x) = \frac{n(k_n+x)}{k_n(n-[nt])} \frac{\mathbb{P}_x(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)}$ and $g(x) = \frac{s+x}{s(1-t)} \frac{p_{1-t}(x, -s)}{p_1(0, -s)}$. But thanks to Gnedenko local limit theorem and the fact that $k_n/a_n \rightarrow s$, we have

$$\lim_{n \rightarrow +\infty} \sup_{x \in [-s, M]} |g_n(x) - g(x)| = 0.$$

Moreover, recall that from Corollary 2.5.1 of Duquesne and Le Gall [9],

$$P^{(n)} \Rightarrow P,$$

as $n \rightarrow +\infty$, where \Rightarrow stands for the weak convergence of measures on \mathbb{D}^2 . Finally, note that the discontinuity set of the functional $F_t D_t I_M(Y_t^1)$ is negligible for the probability measure P so that the last term in (5.23) tends to 0 as n goes to $+\infty$. \blacksquare

Then we will prove the tightness of the sequence, $(X^{br,n}, \bar{H}^{br,n})$. Define the height process associated to any downward skip free chain $x = (x_0, x_1, \dots, x_n, \dots)$, i.e. $x_0 = 0$ and $x_i - x_{i-1} \geq -1$, as follows:

$$H_n^x = \text{card} \left\{ i \in \{0, \dots, n-1\} : x_k = \inf_{i \leq j \leq n} x_j \right\}.$$

Let also $t(k)$ be the first passage time of x by $t(k) = \inf\{i : x_i = -k\}$ and for $n \geq k$, when $t(k) < \infty$, define the shifted chain:

$$\theta_{t(k)}(x)_i = \begin{cases} x_{i+t(k)} + k, & \text{if } i \leq n - t(k) \\ x_{t(k)+i-n} + x_n + k, & \text{is } n - t(k) \leq i \leq n \end{cases}, \quad i = 0, 1, \dots, n,$$

which consists in inverting the pre- $t(k)$ and the post- $t(k)$ parts of x and sticking them together.

Lemma 4. *For any $k \geq 0$, we have almost surely*

$$H^{\theta_{t(k)}(x)} = \theta_{t(k)}(H^x).$$

Proof. It is just a consequence of the fact that $t(k)$ is a zero of H^x . ■

Lemma 5. *Let u_{k_n} be a random variable which is uniformly distributed over $\{0, 1, \dots, k_n\}$ and independent of S . Under $\mathbb{P}(\cdot | T(k_n) = n)$, the first passage time $T(u_{k_n})$ is uniformly distributed over $\{0, 1, \dots, n\}$.*

Proof. It follows from elementary properties of random walks that for all $k \in \{0, 1, \dots, k_n\}$, under $\mathbb{P}(\cdot | T(k_n) = n)$, the chain $\theta_{T(k_n)}(S)$ has the same law as $(S_i, 0 \leq i \leq n)$. As a consequence, for all $j \in \{0, 1, \dots, n\}$,

$$P(T(k) = j | T(k_n) = n) = P(T(k_n - k) = n - j | T(k_n) = n),$$

which allows us to conclude. ■

Lemma 6. *The family of processes*

$$(X^{br,n}, \bar{H}^{br,n}), \quad n \geq 1$$

is tight.

Proof. Let $\mathbb{D}([0, t])$ be the Skorokhod space of càdlàg paths from $[0, t]$ to \mathbb{R} . In Lemma 3 we have proved the weak convergence of $(X^{br,n}, \bar{H}^{br,n})$ restricted to the space $\mathbb{D}([0, t])$ for each $t > 0$. Therefore, from Theorem 15.3 of [4], it suffices to prove that for all $\delta \in (0, 1)$ and $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} \mathbb{P} \left(\sup_{s,t \in [1-\delta, 1]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [1-\delta, 1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right) = 0. \quad (5.24)$$

Recall from Lemma 5 the definition of the r.v. u_{k_n} and define $V_n = \inf\{t : X_t^{br,n} = -\frac{k}{n}\}$. Since from this lemma, V_n is uniformly distributed over $\{0, 1/n, \dots, 1-1/n, 1\}$, we have for any $\varepsilon < 1 - \delta$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{s,t \in [1-\delta, 1]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [1-\delta, 1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right) \leq \varepsilon + \delta + \\ & \mathbb{P} \left(V_n \in [\varepsilon, 1 - \delta], \sup_{s,t \in [1-\delta, 1]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [1-\delta, 1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right). \end{aligned}$$

Now for a càdlàg path ω defined on $[0, 1]$ and $t \in [0, 1]$, define the shift:

$$\theta_t(\omega)_u = \begin{cases} \omega_{s+t+u}, & \text{if } s \leq 1-t \\ \omega_{t+u-1} + \omega_u + k, & \text{is } 1-t \leq s \leq 1 \end{cases}, \quad u \in [0, 1],$$

which consists in inverting the paths $(\omega_u, 0 \leq u \leq t)$ and $(\omega_u, t \leq u \leq 1)$ and sticking them together. We can check on a picture the inclusion:

$$\begin{aligned} & \{V_n \in [\varepsilon, 1 - \delta], \sup_{s,t \in [1-\delta, 1]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [1-\delta, 1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta\} \subset \\ & \left\{ \sup_{s,t \in [0, 1-\varepsilon]} |\theta_{V_n}(X^{br,n})_t - \theta_{V_n}(X^{br,n})_s| > \eta, \sup_{s,t \in [0, 1-\varepsilon]} |\theta_{V_n}(\bar{H}^{br,n})_t - \theta_{V_n}(\bar{H}^{br,n})_s| > \eta \right\}. \end{aligned}$$

From Lemma 4 and the straightforward identity in law $X^{br,n} \stackrel{(d)}{=} \theta_{V_n}(X^{br,n})$, we deduce the two dimensional identity in law $(X^{br,n}, \bar{H}^{br,n}) \stackrel{(d)}{=} (\theta_{V_n}(X^{br,n}), \theta_{V_n}(\bar{H}^{br,n}))$. Hence from the above inequality and inclusion,

$$\begin{aligned} & \mathbb{P} \left(\sup_{s,t \in [1-\delta, 1]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [1-\delta, 1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right) \leq \varepsilon + \delta + \\ & \mathbb{P} \left(\sup_{s,t \in [0, 1-\varepsilon]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [0, 1-\varepsilon]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right). \end{aligned}$$

But from Lemma 3 and Theorem 15.3 in [4], we have

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} \mathbb{P} \left(\sup_{s,t \in [0, 1-\varepsilon]} |X_t^{br,n} - X_s^{br,n}| > \eta, \sup_{s,t \in [0, 1-\varepsilon]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta \right) = 0.$$

which yields (5.24). ■

Proof of Theorem 3. Lemma 3 shows that the sequence of processes $(X^{br,n}, \bar{H}^{br,n})$ converges toward (X^{br}, \bar{H}^{br}) in the sense of finite dimensional distributions. Moreover tightness of this sequence has been proved in Lemma 6, so we conclude from Theorem 15.1 of [4]. The convergence of the two first coordinates in Theorem 3 is proved, i.e. $(X^{br,n}, \bar{H}^{br,n}) \Longrightarrow (X^{br}, \bar{H}^{br})$. Then we may deduce the functional convergence of the third coordinates from this convergence in law following similar arguments as in Theorem 2.4.1 of [9] or in Theorem 3.1 of [8]:

From (2.3), we can recover the contour process of $X^{br,n}$ as follows set $b_i = 2i - \bar{H}_i^{br,n}$, for $0 \leq i \leq n$. For $i \leq n-1$ and $t \in [b_i, b_{i+1})$

$$C_{t/2}^{br,n} = \begin{cases} (\bar{H}_i^{br,n} - (t - b_i))^+ & \text{if } t \in [b_i, b_{i+1} - 1), \\ (\bar{H}_{i+1}^{br,n} - (b_{i+1} - t))^+, & \text{if } t \in [b_{i+1} - 1, b_{i+1}), \end{cases}.$$

Hence for $0 \leq i \leq n-1$,

$$\sup_{b_i \leq t < b_{i+1}} \left| C_{t/2}^{br,n} - \bar{H}_i^{br,n} \right| \leq \left| \bar{H}_{i+1}^{br,n} - \bar{H}_i^{br,n} \right| + 1. \quad (5.25)$$

Now, we define $h_n(t) = i$, if $t \in [b_i, b_{i+1})$ and $i \leq n-1$, and $h_n(t) = n$, if $t \in [2n-2, 2n]$. The definitions of b_i and h_n implies

$$\sup_{0 \leq t \leq 2n} \left| h_n(t) - \frac{t}{2} \right| \leq \frac{1}{2} \sup_{0 \leq k \leq n} \bar{H}_k^{br,n} + 1.$$

Next, we set $f_n(t) = h_n(nt)/n$. By (5.25), we have

$$\sup_{0 \leq t \leq 2} \frac{a_n}{n} \left| C_{nt/2}^{br,n} - \bar{H}_{nf_n(t)}^{br,n} \right| \leq \frac{a_n}{n} \sup_{0 \leq t \leq 1} \left| \bar{H}_{[nt]+1}^{br,n} - \bar{H}_{[nt]}^{br,n} \right| + \frac{a_n}{n},$$

and

$$\sup_{0 \leq t \leq 2} \left| f_n(t) - \frac{t}{2} \right| \leq \frac{1}{2a_n} \sup_{0 \leq k \leq n} \frac{a_n}{n} \bar{H}_k^{br,n} + \frac{1}{p}.$$

From our hypothesis, we get

$$\frac{a_n}{n} \sup_{0 \leq t \leq 1} \left| \bar{H}_{[nt]+1}^{br,n} - \bar{H}_{[nt]}^{br,n} \right| + \frac{a_n}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$\frac{1}{2a_n} \sup_{0 \leq k \leq n} \frac{a_n}{n} \bar{H}_k^{br,n} + \frac{1}{p} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

in probability. Hence, from the convergence of $(X^{br,n}, \bar{H}^{br,n})$ to (X^{br}, \bar{H}^{br}) , from Theorem 4.1 in [4] and Skorokhod representation theorem we obtain the convergence,

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n}) \Rightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br}).$$

■

Remarks: By a classical time reversal argument, the weak convergence of the first coordinate in Theorem 3 implies the main result of Bryn-Jones and R.A. Doney [5]. Indeed, when X is the standard Brownian motion, it is well known that the returned first passage bridge $(s + X_{1-u}^{br}, 0 \leq u \leq 1)$ is the bridge of a three dimensional Bessel process from 0 to s with length 1. Similarly, the returned discrete first passage bridge whose law is this of $(k_n + S_{n-i}, 0 \leq i \leq n)$ under $\mathbb{P}(\cdot | T(k_n) = n)$ has the same law as $(S_i, 0 \leq i \leq n)$ given $S_n = k_n$ and conditioned to stay positive. Then integrating with respect to the terminal values and applying Theorem 3 gives the result contained in [5].

References

- [1] D. ALDOUS: The continuum random tree. I. *Ann. Probab.*, **19**, No.1, 1-28, (1991).
- [2] J. BERTOIN (1996): *Lévy Processes*. Cambridge University Press, Cambridge.
- [3] J. BERTOIN, L. CHAUMONT AND J. PITMAN (2003): Path transformations of first passage bridges. *Elect. Comm. in Probab.* **8**, 155–166.
- [4] P. BILLINGSLEY: Convergence of probability measures. Second edition. *John Wiley & Sons, Inc.*, New York, 1999.
- [5] A. BRYN-JONES AND R.A. DONEY: A functional limit theorem for random walk conditioned to stay non-negative. *J. London Math. Soc.* (2), **74**, no. 1, 244–258, (2006).
- [6] L. CHAUMONT AND G. URIBE: Bridges of self-similar Markov processes. *Work in progress*.
- [7] A. DRESS, V. MOULTON AND W. TERHALLE: T -theory: an overview. *European J. Combin.* **17**, no. 2-3, 161–175, (1996).

- [8] T. DUQUESNE: A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.*, **31**, No.2, 996-1027 (2003).
- [9] T. DUQUESNE AND J.F. LE GALL: Random trees, Lévy processes and spatial branching processes. *Astérisque* 281. Paris: Société Mathématique de France. vi, (2002).
- [10] T. DUQUESNE AND J.F. LE GALL: The Hausdorff measure of stable trees. *ALEA Lat. Am. J. Probab.*, Math. Stat. **1**, 393–415, (2006).
- [11] T. DUQUESNE AND J.F. LE GALL: Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields*, **131**, no. 4, 553–603, (2005).
- [12] S. EVANS: Probability and trees. *Séminaire de Probabilités XXXVIII*, 1–4, Lecture notes in Math., 1857, Springer, Berlin, (2005).
- [13] S. EVANS, J. PITMAN AND A. WINTER: Rayleigh processes, real trees and root growth with re-grafting. *Probab. Theory Related Fields*, **134**, no 1, 81–126, (2006).
- [14] W. FELLER *An introduction to probability theory and its applications. Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney*, 1971.
- [15] P.J. FITZSIMMONS, J. PITMAN AND M. YOR (1993): Markovian bridges: construction, Palm interpretation, and splicing. *Seminar on Stochastic Processes*, 1992, 101–134, Progr. Probab., 33, Birkhäuser Boston, MA.
- [16] A. GRIMVALL: On the convergence of sequences of branching processes. *Ann. Probab.*, **2**, 1027-1045 (1974).
- [17] M. JIRINA: Stochastic branching processes with continuous state-space. *Czech. Math. J.*, **8**, 292-313, (1958).
- [18] J. LAMPERTI: The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **7**, 271–288, (1967).
- [19] J. LAMPERTI: Limiting distributions of branching processes. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.*, **2**, 225–241. University California Press, Berkeley. (1967).
- [20] J.F. LE GALL: Random real trees. *Ann. Fac. Sci. Toulouse Math.*, (6), **15**, no. 1, 35–62, (2006).
- [21] J.F. LE GALL AND Y. LE JAN: Branching processes in Lévy processes: The exploration process. *Ann. Probab.*, **26**, No.1, 213–252, (1998).
- [22] J. PITMAN: Combinatorial stochastic processes. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002. Lecture Notes in Mathematics, 1875. Springer-Verlag, Berlin, 2006
- [23] J. PITMAN AND M. WINKEL: Growth of the Brownian forest. *Ann. Probab.*, **33**, no. 6, 2188–2211, (2005).